

DERIVATION OF THE LINEAR LANDAU EQUATION AND LINEAR BOLTZMANN EQUATION FROM THE LORENTZ MODEL WITH MAGNETIC FIELD

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ABSTRACT. We consider a test particle moving in a random distribution of obstacles in the plane, under the action of a uniform magnetic field, orthogonal to the plane. We show that, in a weak coupling limit, the particle distribution behaves according to the linear Landau equation with a magnetic transport term. Moreover, we show that, in a low density regime, when each obstacle generates an inverse power law potential, the particle distribution behaves according to the linear Boltzmann equation with a magnetic transport term. We provide an explicit control of the error in the kinetic limit by estimating the contributions of the configurations which prevent the Markovianity. We compare these results with those ones obtained for a system of hard disks in [BMHH], which show instead that the memory effects are not negligible in the Boltzmann-Grad limit.

Keywords: Lorentz gas; magnetic field; linear Boltzmann equation; linear Landau equation; low density limit; weak coupling limit.

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1. INTRODUCTION

Consider a point particle of mass $m = 1$ in \mathbb{R}^d , $d = 2, 3$ moving in a random distribution of fixed scatterers, whose centers are denoted by (c_1, \dots, c_N) .

We assume that the scatterers are distributed according to a Poisson distribution of parameter $\mu > 0$. The equations of motion are

$$(1.1) \quad \begin{cases} \dot{x} = v \\ \dot{v} = -\sum_i \nabla \phi(|x - c_i|) \end{cases} ,$$

here (x, v) denote position and velocity of the test particle, t the time and $\dot{A} = \frac{dA}{dt}$ for any time dependent variable A .

To outline a kinetic behavior it is usually introduced a scaling of the space-time variables and the density of the scatterer distribution. For this model, it is more physically intuitive to transfer the scaling to the background medium. More precisely, let $\varepsilon > 0$ be a parameter indicating the ratio between

the macroscopic and microscopic variables, we keep time and space fixed and rescale the range of the interaction and the density of the scatterers, i.e.

$$(1.2) \quad \begin{aligned} \phi_\varepsilon(x) &= \varepsilon^\alpha \phi\left(\frac{x}{\varepsilon}\right) \\ \mu_\varepsilon &= \mu \varepsilon^{-(d-1+2\alpha)} \end{aligned}$$

where $d = 2, 3$ is the dimension of the physical space and $\alpha \in [0, \frac{1}{2}]$ is a suitable parameter. This means that the probability of finding N obstacles in a bounded measurable set $\Lambda \subset \mathbb{R}^d$ is given by

$$(1.3) \quad \mathbb{P}_\varepsilon(d\mathbf{c}_N) = e^{-\mu_\varepsilon|\Lambda|} \frac{\mu_\varepsilon^N}{N!} dc_1 \cdots dc_N$$

where $\mathbf{c}_N = c_1, \dots, c_N$ and $|\Lambda| = \text{meas}(\Lambda)$. Consequently, the equation of motion (1.1) becomes

$$(1.4) \quad \begin{cases} \dot{x} = v \\ \dot{v} = -\varepsilon^{\alpha-1} \sum_i \nabla \phi\left(\frac{|x-c_i|}{\varepsilon}\right) \end{cases}.$$

Now let $T_{\mathbf{c}_N}^t(x, v)$ be the Hamiltonian flow solution to equation (1.4) with initial datum (x, v) in a given sample of obstacles (skipping the ε dependence for notational simplicity) and, for a given probability distribution $f_0 = f_0(x, v)$, consider the quantity

$$(1.5) \quad f_\varepsilon(x, v, t) = \mathbb{E}_\varepsilon[f_0(T_{\mathbf{c}_N}^{-t}(x, v))]$$

where \mathbb{E}_ε is the expectation with respect to the measure \mathbb{P}_ε given by (1.3).

In the limit $\varepsilon \rightarrow 0$ we expect that the probability distribution (1.5) will solve a linear kinetic equation depending on the value of α . If $\alpha = 0$ the limit corresponding to such a scaling is called low-density (or Boltzmann-Grad) limit. In this case f_ε converges to the solution of a linear Boltzmann equation. See [G, S, BBS, DP]. On the other hand, if $\alpha = \frac{1}{2}$ the corresponding limit, called weak-coupling limit, yields the linear Landau equation, as proven in [KP, DGL, K]. The intermediate scaling, namely $\alpha \in (0, \frac{1}{2})$, although refers to a low-density regime, leads to the linear Landau equation again, see [DR, K].

We want to remark that in [DP, DR] the authors exploit the original constructive idea due to Gallavotti (see [G]) for the Boltzmann-Grad limit. This method is based on a suitable change of variables which can be implemented outside a set E of pathological events which prevent the Markov property of the limit (such as the set of configurations yielding recollisions, i.e. when the test particle recollides with a given obstacle after having suffered collisions with other different obstacles). The probability $\mathbb{P}_\varepsilon(E)$ is vanishing as ε tends to 0. The main difference is that in [DP] the range of the potential is infinite in the limit, therefore the test particle interacts with infinitely many obstacles. As for the case of the long range potential considered in [DP] also in [DR] there is a lack of the semi explicit form of the solution of the limit equation. This requires explicit estimates for the set of bad configurations of obstacles. For a short range potential, like in the case of the hard-sphere potential considered in [G], a simple dimensional argument is sufficient. For an explicit control of the error in the kinetic limit for the hard-sphere potential see for instance [BNPP]. Moreover, in [DR, K] it was proven that even if $\alpha > 0$, but sufficiently small, the recollisions are still negligible. Incidentally we note that, if α is close to $1/2$, this is not true anymore and it would be interesting to derive the Landau equation in this regime, by means of an explicit constructive approach.

Furthermore it has been observed that the presence of a given external field, in the two dimensional Lorentz model, strongly affects the derivation of the linear Boltzmann equation in the Boltzmann-Grad limit. Bobylev et al, in [BMHH] and later in [BMHH1, BHPH] (see also [KS] for further readings), showed that the set of pathological configurations is no longer negligible when the test particle moves in a plane with a Poisson distribution of hard disks and a uniform and constant magnetic field perpendicular to the plane. See Figure 1 for a pictorial representation of the light particle's motion.

The following simple computation turns out to give a good heuristic argument explaining these results: consider the probability \mathbb{P}_{R_L} of performing an entire Larmor circle without hitting any obstacle, R_L being the Larmor radius. From equation (1.3) one easily gets

$$\mathbb{P}_{R_L} \simeq e^{-\mu_\varepsilon \text{Area}(\mathcal{A}_\varepsilon)} \simeq e^{-2\pi R_L \mu},$$

where $\mathcal{A}_\varepsilon(R_L)$ is the annulus of radius R_L and width ε . Hence, \mathbb{P}_{R_L} is not vanishing in the limit $\varepsilon \rightarrow 0$ and the Markovianity of the limit system can not be attained. In fact, in [BMHH1, BHPH], a kinetic

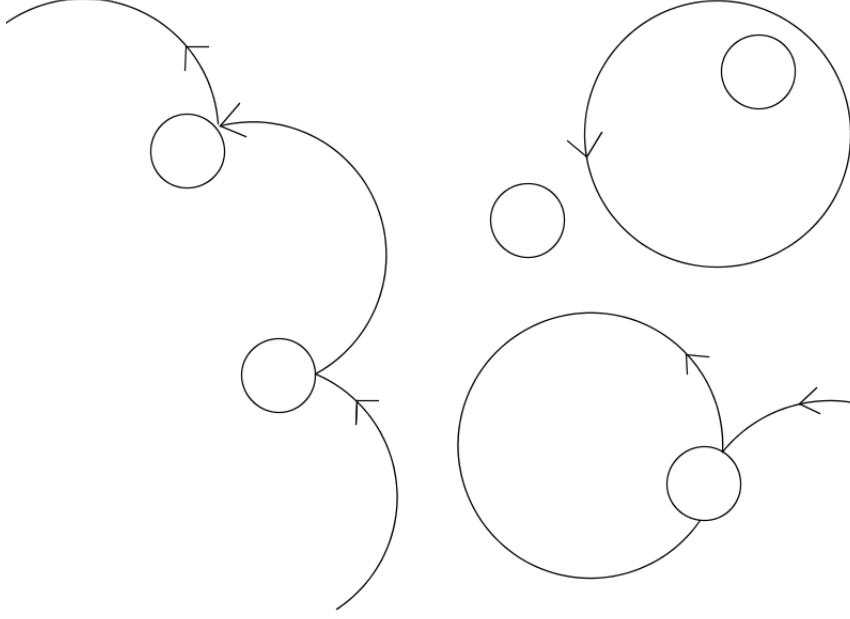


FIGURE 1. Typical paths of the charged test particle when the obstacles are hard disks: due to the magnetic field, it performs arcs of circle between two consecutive collisions.

equation with memory is derived, i.e. a generalized Boltzmann equation, taking into account those effects:

$$(1.6) \quad \frac{D}{Dt} f^G(x, v, t) = \mu_\varepsilon \varepsilon \sum_{k=0}^{\lfloor t/T_L \rfloor} e^{-\nu k T_L} \int_{S_1} dn (v \cdot n) [\chi(v \cdot n) b_n + \chi(-v \cdot n)] f^G(x, S_0^{-k} v, t - k T_L),$$

where $f(x, v, t)$ is the probability density of finding the moving particle at time t at position x with velocity v and

$$(1.7) \quad f^G(x, v, t) = \begin{cases} f(x, v, t) & \text{if } 0 < t < T_L \\ (1 - e^{-\nu T_L}) f(x, v, t) & \text{if } t > T_L. \end{cases}$$

Here $\nu = 2|v|\mu_\varepsilon \varepsilon$ is the collision frequency and $T_L = 2\pi/\Omega$ is the cyclotron period where $\Omega = qB/m$ is the frequency, being q the charge and m the mass. Furthermore, note that

$$\frac{D}{Dt} = (\partial_t + v \cdot \nabla_x + (v \times B) \cdot \nabla_v)$$

is the generator of the free cyclotron motion with frequency Ω and $\lfloor t/T_L \rfloor$ the number of cyclotron periods T_L completed before time t . The angular integration over the unit vector n in (1.6) is over the entire unit sphere S_1 centered at the origin. In the gain term the operator b_n is defined by

$$b_n \phi(v) = \phi(v - 2(v \cdot n)n)$$

where $\phi(v)$ is an arbitrary function of v . The precollisional velocity $v' = v - 2(v \cdot n)n$ becomes v after the elastic collision with the hard disk. Note that $v' \cdot n < 0$. In the loss term, the precollisional velocity v is also from the hemisphere $v \cdot n < 0$. Finally, the shift operator S_0^{-k} , when acting on v , rotates the velocity through the angle $-k\theta$, where θ is the scattering angle (from v' to v).

For further readings in this direction we refer to [DR1, DR2], where the authors consider a stochastic Lorentz model with a smooth external force field $F(x, t)$ and with absorbing obstacles, i.e. the interaction between the obstacles and the test particle is such that the test particle disappears whenever it enters an obstacle. It is proved that the kinetic equation associated to this model in the Boltzmann-Grad limit is non-Markovian and that the Markovianity can be recovered by introducing an additional stochasticity in the velocity distribution of the obstacles.

In this paper we consider the case of a random distribution of scatterers in \mathbb{R}^2 where each obstacle generates a smooth positive and short-range potential ϕ , with $\alpha > 0$ and sufficiently small. We show that, in this case, the solution of the microscopic dynamics converges, in the intermediate limit (when $\alpha \in (0, 1/8)$), to the solution of the linear Landau equation with an additional transport term due to the magnetic field. From the heuristic point of view, this result is suggested by the observation that in this case the probability \mathbb{P}_{R_L} of performing an entire Larmor circle without hitting any obstacle is given by

$$\mathbb{P}_{R_L} \simeq e^{-\mu_\varepsilon 2\pi \varepsilon R_L} \simeq e^{-2\pi R_L \mu \varepsilon^{-2\alpha}}$$

which vanishes as $\varepsilon \rightarrow 0$. This computation shows that one family of the pathological events preventing the Markovianity is negligible in this setting. We stress that this rough argument is not sufficient to conclude that we can recover the Markovianity in the limit. Indeed, to prove this, we need to show that all the other bad configurations of obstacles defining the set E are negligible in the limit, as we will see in Section 4.1.

Furthermore, we observe that even if we consider a long range inverse power law interaction potential, truncated at distance $\varepsilon^{\gamma-1}$ with $\gamma \in (0, 1)$ suitably large, in the low density regime $\alpha = 0$, we can prove that the memory is lost in the limit. More precisely, we prove that the microscopic solution converges to the solution of the uncutted linear Boltzmann equation with a magnetic transport term. With the same purpose of the rough argument presented above, we observe that the probability \mathbb{P}_{R_L} of performing a complete Larmor circle without hitting any obstacle is approximatively given by

$$\mathbb{P}_{R_L} \simeq e^{-2\pi R_L \mu \varepsilon^{\gamma-1}}$$

which vanishes as $\varepsilon \rightarrow 0$ when $\gamma < 1$. Also in this case this represents only one example of bad configuration of scatterers. It is essential to prove that the contribution of the whole set of pathological events is negligible in the limit, as we will show in Section 5. Moreover, from the technical point of view, we observe that the parameter γ has to be chosen close to 1 as dictated by the explicit control of the memory effects.

Thus, as we pointed out with the heuristic motivations above, the non-Markovian behaviour of the limit process, discussed in [BMHH], disappears as soon as we slightly modify the microscopic model given by the two dimensional Lorentz Gas.

The purpose of this paper is to provide a rigorous validation of the linear Landau equation and the linear Boltzmann equation respectively with magnetic field by using the constructive strategy due to Gallavotti. We remark that, as in [DP, DR, BNP, BNPP], we need explicit estimates of the error in the kinetic limit and this is the crucial part. Moreover, as a future target, it could be interesting to understand if a rigorous derivation of the generalized Boltzmann equation proposed in [BMHH] can be achieved by using the same constructive techniques.

The plan of the paper is the following: in the next Section we establish the model and formulate the results; in Section 3 we present the strategy of the proofs, whereas Sections 4 and 5 are dedicated to the nontrivial analysis and explicit estimates of the sets of bad configurations producing memory effects, which is the technical core of this paper.

2. THE MODEL AND MAIN RESULTS

2.1. The Lorentz Model with short range interactions. We consider the system (1.4) in the plane ($d = 2$) under the action of a uniform, constant, magnetic field orthogonal to the plane. The equations of motion are

$$(2.1) \quad \begin{cases} \dot{x} = v \\ \dot{v} = Bv^\perp - \varepsilon^{\alpha-1} \sum_i \nabla \phi\left(\frac{|x-c_i|}{\varepsilon}\right) \end{cases},$$

where B is the magnitude of the magnetic field and $v^\perp = (v_2, -v_1)$. We assume that the potential $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is smooth and of range 1 i.e. $\phi(r) = 0$ if $r > 1$. Therefore the particle is influenced by the scatterer c_i if $|x - c_i| < \varepsilon$.

Starting from the initial position x with initial velocity v , the particle moves under the action of the Lorentz force Bv^\perp . Suppose that the particle has unitary mass and unitary charge, namely $m, q = 1$, hence between two consecutive scatterers, the particle moves with constant angular velocity $\Omega = qB/m = B$ and performs an arc of circle of radius $R_L = |v|/B$. R_L is the Larmor radius, i.e. the radius of the

cyclotron orbit whose center is situated at the point

$$x_c = x + \frac{\mathcal{R}(\frac{\pi}{2}) \cdot v}{\Omega},$$

where the tensor $\mathcal{R}(\varphi)$ denotes the rotation of angle φ . Without loss of generality we assume from now on that $|v| = 1$, therefore $R_L = 1/B$. Moreover, we will denote by S_1 the kinetic energy sphere with unitary radius.

The precise assumptions on the potential are the following:

- A1) $\phi \in C^2([0, \infty))$;
- A2) $\phi \geq 0$, $\phi' \leq 0$ in $(0, 1)$;
- A3) $\text{supp } \phi \subset [0, 1]$.

On f_0 we assume that

- A4) $f_0 \in C_0(\mathbb{R}^2 \times \mathbb{R}^2)$ is a continuous, compactly supported initial probability density. Suppose also that $|D_x^k f_0| \leq C$, where D_x is any partial derivative with respect to x and $k = 1, 2$.

Moreover, we assume that

- A5) The scatterers are distributed according to a Poisson distribution (1.3) of intensity $\mu_\varepsilon = \mu \varepsilon^{-\delta}$ with $\delta = 1 + 2\alpha$, $\alpha \in (0, \frac{1}{8})$.

Next we define the Hamiltonian flow $T_{\mathbf{c}_N}^t(x, v)$ associated to the initial datum (x, v) , solution of (2.1) for a given configuration \mathbf{c}_N of scatterers, and we set

$$(2.2) \quad f_\varepsilon(x, v, t) = \mathbb{E}_\varepsilon[f_0(T_{\mathbf{c}_N}^{-t}(x, v))]$$

where \mathbb{E}_ε denotes the expectation with respect to the Poisson distribution.

The first result of the present paper is summarized in the following theorem.

Theorem 2.1. *Let f_ε be defined in (2.2). Under assumption A1) – A5), for all $t \in [0, T]$, $T > 0$,*

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(\cdot; t) = g(\cdot; t)$$

where g is the unique solution to the Landau equation with magnetic field

$$(2.3) \quad \begin{cases} (\partial_t + v \cdot \nabla_x + B v^\perp \cdot \nabla_v) g(x, v, t) = \xi \Delta_{S_1} g(x, v, t) \\ g(x, v, 0) = f_0(x, v) \end{cases},$$

where Δ_{S_1} is the Laplace-Beltrami operator on the circle S_1 and $\xi > 0$. The convergence is in $L^2(\mathbb{R}^2 \times S_1)$.

The constant ξ is the diffusion coefficient and its explicit expression will be given below in (3.9) and in remark 3.4.

2.2. The Lorentz Model with long range interactions. We consider now the case in which each obstacle generates a potential of the form

$$\check{\psi}_\varepsilon(|x - c|) = \psi_\varepsilon\left(\frac{|x - c|}{\varepsilon}\right)$$

where the unrescaled potential ψ_ε is an inverse power law potential truncated at large distances. More precisely we assume the following:

$$B1) \quad \psi_\varepsilon(x) = \begin{cases} \frac{1}{|x|^s} & |x| < \varepsilon^{\gamma-1} \\ \varepsilon^{-s(\gamma-1)} & |x| \geq \varepsilon^{\gamma-1} \end{cases} \quad \text{with } \gamma \in (0, 1) \text{ and } s > 2.$$

We point out that it could be challenging to consider directly the untruncated long range potential $\psi(|x|) = |x|^{-s}$. In fact, this problem presents deep additional difficulties as noted in Remark 2.3 in [DP] and new ideas and techniques are necessary.

Moreover, we assume that

- B2) The scatterers are distributed according to a Poisson law (1.3) of intensity $\mu_\varepsilon = \varepsilon^{-1} \mu$, $\mu > 0$.

The equation of motion in macroscopic variables reads

$$(2.4) \quad \begin{cases} \dot{x} = v \\ \dot{v} = B v^\perp - \varepsilon^{-1} \sum_i \nabla \psi_\varepsilon\left(\frac{|x - c_i|}{\varepsilon}\right) \end{cases}$$

with ψ_ε given in Assumption B1).

Let $T_{\mathbf{c}_N}^t(x, v)$ be the Hamiltonian flow solution to equation (2.4) with initial datum (x, v) in a given sample of obstacles. Let be $f_0 = f_0(x, v)$ be the initial probability distribution. On f_0 we assume

$$B3) \quad f_0 \in L^1 \cap W^{1,\infty}(\mathbb{R}^2 \times \mathbb{R}^2), \quad f_0 \geq 0, \quad \int f_0 dx dv = 1.$$

We consider the quantity

$$(2.5) \quad f_\varepsilon(x, v, t) = \mathbb{E}_\varepsilon[f_0(T_{\mathbf{e}_N}^{-t}(x, v))]$$

where \mathbb{E}_ε is the expectation with respect to the measure \mathbb{P}_ε given by (1.3). The second result of this paper is summarized in the following theorem.

Theorem 2.2. *Let f_ε be defined in (2.5). Under assumption B1) – B3) with $\gamma \in (6/7, 1)$, for all $t \in [0, T]$, $T > 0$,*

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(\cdot; t) = f(\cdot; t)$$

where f is the unique solution to the linear Boltzmann equation with magnetic field

$$(2.6) \quad \begin{cases} (\partial_t + v \cdot \nabla_x + B v^\perp \cdot \nabla_v) f(t, x, v) = Lf(t, x, v) \\ f(x, v, 0) = f_0(x, v), \end{cases}$$

with

$$Lf(v) = \mu \int_{-\pi}^{\pi} \Gamma(\Theta) \{f(\mathcal{R}(\Theta)v) - f(v)\} d\Theta,$$

$\Gamma(\Theta)$ is the differential cross section associated to the long range potential $\psi(|x|) = |x|^{-s}$ and the operator $\mathcal{R}(\Theta)$ rotates the velocity v by the angle Θ . The convergence is in $\mathcal{D}'(\mathbb{R}^2 \times S_1)$.

3. PROOFS

3.1. Proof of Theorem 2.1. Following [DP, DR, BNP] we split the original problem into two parts. The first one concerns the asymptotic equivalence between f_ε defined in (1.5) and h_ε , solution of the following Boltzmann equation

$$(3.1) \quad \begin{cases} (\partial_t + v \cdot \nabla_x + B v^\perp \cdot \nabla_v) h_\varepsilon(x, v, t) = L_\varepsilon h_\varepsilon(x, v, t) \\ h_\varepsilon(x, v, 0) = f_0(x, v) \end{cases}$$

where

$$(3.2) \quad L_\varepsilon h_\varepsilon(v) = \mu_\varepsilon \int_{-\varepsilon}^{\varepsilon} d\rho [h_\varepsilon(v') - h_\varepsilon(v)].$$

Here $v' = v - 2(\omega \cdot v)\omega$ is the outgoing velocity after a scattering with incoming velocity v and impact parameter $\rho \in [-\varepsilon, \varepsilon]$ generated by the potential $\varepsilon^\alpha \phi(\frac{r}{\varepsilon})$. Moreover, $\omega = \omega(\rho)$ is the versor bisecting the angle between the incoming and outgoing velocity and θ_ε is the scattering angle. The precise result is stated in the following proposition.

Proposition 3.1. *Under assumption A1) – A5), for any $T > 0$,*

$$(3.3) \quad \lim_{\varepsilon \rightarrow 0} \|f_\varepsilon - h_\varepsilon\|_{L^\infty([0, T]; L^1(\mathbb{R}^2 \times S_1))} = 0$$

where h_ε solves (3.1).

The proof of the above Proposition is postponed to Section 4.

The second step concerns the grazing collision limit. Note that the presence of the magnetic field does not affect the last step. More precisely we have the following

Lemma 3.2. *The deflection angle $\theta_\varepsilon(\rho)$ of a particle colliding with impact parameter ρ with a scatterer generating a radial potential $\varepsilon^\alpha \phi$ under the action of the Lorentz force Bv^\perp satisfies*

$$(3.4) \quad |\theta_\varepsilon(\rho)| \leq C\varepsilon^\alpha.$$

Proof. As established in [DR] (Section 3), the estimate (3.4) holds when the test particle scatters with no external field. Hence, we just need to compare the dynamics of the test particle in presence of the constant magnetic field with the free dynamics. Let $(\underline{x}(t), \underline{v}(t))$ be the solution of the following

$$(3.5) \quad \begin{cases} \dot{\underline{x}} = \underline{v} \\ \dot{\underline{v}} = -\varepsilon^{\alpha-1} \nabla \phi(\frac{|\underline{x}-c|}{\varepsilon}) \end{cases}.$$

Let τ be the collision time for the dynamics described by (2.1). The key observation is that the presence of the magnetic field does not modify the estimate for the collision time related to the dynamics in (3.5).

Indeed also in our case $\tau \leq C\varepsilon$, $C > 0$, as in [DR], see Appendix A for the detailed computations. Therefore we get

$$\begin{aligned}
|v(\tau) - \underline{v}(\tau)| &= \left| \varepsilon^{\alpha-1} \int_0^\tau ds (F(x(s)/\varepsilon) - F(\underline{x}(s)/\varepsilon)) + \int_0^\tau ds v^\perp B \right| \\
&\leq \varepsilon^{\alpha-1} \int_0^\tau ds |F(x(s)/\varepsilon) - F(\underline{x}(s)/\varepsilon)| + C_1 \varepsilon \\
&\leq \varepsilon^{\alpha-2} C_2 \int_0^\tau ds |x(s) - \underline{x}(s)| + C_1 \varepsilon \\
&\leq \varepsilon^{\alpha-2} C_2 \int_0^\tau ds \int_0^s dt |v(t) - \underline{v}(t)| + C_1 \varepsilon \\
&\leq \varepsilon^{\alpha-2} C_2 \int_0^\tau ds \int_0^\tau dt |v(t) - \underline{v}(t)| + C_1 \varepsilon,
\end{aligned}$$

where $F(x) := -(\nabla\phi)(x)$. By using Grönwall's inequality we obtain

$$(3.6) \quad |v(\tau) - \underline{v}(\tau)| \leq C_1 \varepsilon e^{C_3 \varepsilon^{\alpha-1} \tau} \leq C_1 \varepsilon e^{C_3 \varepsilon^\alpha}$$

for $\alpha > 0$ and ε sufficiently small. Hence, the velocities v and \underline{v} are asymptotically equivalent up to an error term of order ε . We now define v' and \underline{v}' to be the outgoing velocities with and without magnetic field respectively, v the incoming velocity. By using (3.6) we have

$$(3.7) \quad \sqrt{2(1 - \cos \theta_\varepsilon)} = |v' - v| \leq |v' - \underline{v}'| + |\underline{v}' - v| \leq C\varepsilon + \sqrt{2(1 - \cos \tilde{\theta}_\varepsilon)}$$

where $\tilde{\theta}_\varepsilon$ is the scattering angle without magnetic field. From [DR] we know that $\tilde{\theta}_\varepsilon \leq C'\varepsilon^\alpha$, so from (3.7) we obtain

$$|\sin \frac{\theta_\varepsilon}{2}| \leq C\varepsilon + |\sin \frac{\tilde{\theta}_\varepsilon}{2}| \leq C\varepsilon + C'\varepsilon^\alpha \leq C''\varepsilon^\alpha.$$

Since θ_ε is continuous as a function of the impact parameter ρ , it results $\theta_\varepsilon \leq C\varepsilon^\alpha$. For further details see Proposition B.1 in Appendix B. \square

The following proposition shows the asymptotic equivalence between the solution of Landau equation and the solution of the previous Boltzmann equation h_ε .

Proposition 3.3. *Under the assumptions A1) – A5), $h_\varepsilon \rightarrow g$ in $L^\infty([0, T]; L^2(\mathbb{R}^2 \times S_1))$ where g is the unique solution to the Landau equation with magnetic field*

$$(3.8) \quad \begin{cases} (\partial_t + v \cdot \nabla_x + B v^\perp \cdot \nabla_v) g(x, v, t) = \xi \Delta_{S_1} g(x, v, t) \\ g(x, v, 0) = f_0(x, v) \end{cases},$$

where

$$(3.9) \quad \xi = \lim_{\varepsilon \rightarrow 0} \frac{\mu \varepsilon^{-2\alpha}}{2} \int_{-1}^1 \theta_\varepsilon^2(\rho) d\rho$$

is the diffusion coefficient.

Remark 3.4. *As shown in Appendix B, $\theta_\varepsilon = \tilde{\theta}_\varepsilon + O(\varepsilon)$, where $\tilde{\theta}_\varepsilon$ is the scattering angle without any magnetic field, i.e. the scattering angle studied in [DR]. This implies that the explicit expression for the diffusion coefficient obtained in [DR] still holds in our case:*

$$(3.10) \quad \xi = \frac{\mu}{2} \int_{-1}^1 \left(\int_\rho^1 \frac{\rho}{u} \phi'(\frac{\rho}{u}) \frac{du}{\sqrt{1-u^2}} \right)^2 d\rho,$$

where the integrand is an even function of the impact parameter ρ .

Remark 3.5. *The linear Landau equation (3.8) propagates the regularity of the derivatives with respect to the x variable thanks to the transport operator. Moreover, the presence of the collision operator $\mathcal{L} := \Delta_{S_1}$ lets the solution gain regularity with respect to the transverse component of the velocity. Indeed, under the assumption A4) on f_0 , the solution $g : \mathbb{R}^2 \times S_1 \rightarrow \mathbb{R}^+$ satisfies the bounds*

$$(3.11) \quad |D_x^k g| \leq C, \quad |D_v^h g(x, v)| \leq C \quad \forall k \leq 2, h \geq 0,$$

$\forall t \in (0, T]$, where $C = C(f_0, T)$ and D_v is the derivative with respect to the transverse component of the velocity. In particular, the solutions of (3.8) we are considering are classical.

Proof. By using the invariance of the scattering angle with respect to the space scale, we rewrite the collision operator in the right hand side of (3.1) as

$$(3.12) \quad L_\varepsilon h_\varepsilon(v) = \mu_\varepsilon \varepsilon \int_{-1}^1 d\rho [h_\varepsilon(v') - h_\varepsilon(v)].$$

We look at the evolution of $h_\varepsilon - g$, being g the solution of (3.8), namely

$$(3.13) \quad (\partial_t + v \cdot \nabla_x + B v^\perp \cdot \nabla_v)(h_\varepsilon - g) = (L_\varepsilon h_\varepsilon - \mathcal{L}g),$$

where $\mathcal{L} := \xi \Delta_{S_1}$.

Note that $g \in L^2(\mathbb{R}^2 \times S_1)$ because $f_0 \in L^2(\mathbb{R}^2 \times \mathbb{R}^2)$ and $h_\varepsilon \in L^2(\mathbb{R}^2 \times S_1)$. Indeed, from Proposition 3.1, we know that $h_\varepsilon \in L^1(\mathbb{R}^2 \times S_1)$ but the hypothesis on the initial state implies that $h_\varepsilon \in L^2(\mathbb{R}^2 \times S_1)$.

We now consider the scalar product of equation (3.13) with $(h_\varepsilon - g)$ in $L^2(\mathbb{R}^2 \times S_1)$ and we obtain

$$\begin{aligned} \frac{1}{2} \partial_t \|h_\varepsilon - g\|_2^2 &= - (h_\varepsilon - g, -L_\varepsilon [h_\varepsilon - g]) \\ &\quad + (h_\varepsilon - g, [L_\varepsilon - \mathcal{L}]g). \end{aligned}$$

By exploiting the positivity of $-L_\varepsilon$ and the Cauchy-Schwartz inequality we get

$$\partial_t \|h_\varepsilon - g\|_2 \leq \| (L_\varepsilon - \mathcal{L})g \|_2.$$

We now set

$$\begin{aligned} g(v') - g(v) &= (v' - v) \cdot \nabla_{|S_1} g(v) \\ &\quad + \frac{1}{2} (v' - v) \otimes (v' - v) \nabla_{|S_1} \nabla_{|S_1} g(v) \\ &\quad + \frac{1}{6} (v' - v) \otimes (v' - v) \otimes (v' - v) \nabla_{|S_1} \nabla_{|S_1} \nabla_{|S_1} g(v) + R_\varepsilon, \end{aligned}$$

with $R_\varepsilon = \mathcal{O}(|v - v'|^4)$. Integrating with respect to ρ and using symmetry arguments we obtain

$$L_\varepsilon g = \mu \varepsilon^{-2\alpha} \left\{ \frac{1}{2} \Delta_{S_1} g \int_{-1}^1 d\rho |v' - v|^2 + \int_{-1}^1 d\rho R_\varepsilon \right\}.$$

Observe that $|v' - v|^2 = 4 \sin^2 \frac{\theta_\varepsilon(\rho)}{2}$, then by direct computation

$$\lim_{\varepsilon \rightarrow 0} \frac{\mu \varepsilon^{-2\alpha}}{2} \int_{-1}^1 d\rho |v' - v|^2 = \lim_{\varepsilon \rightarrow 0} \frac{\mu \varepsilon^{-2\alpha}}{2} \int_{-1}^1 \theta_\varepsilon^2(\rho) d\rho =: \xi.$$

Therefore, thanks to Lemma 3.2, we have

$$\| (L_\varepsilon - \mathcal{L})g \|_{L^2} \leq \varepsilon^{2\alpha} \|\Delta_{|S_1}^2 g\|_{L^2} \leq \varepsilon^{2\alpha} C,$$

which vanishes for $\varepsilon \rightarrow 0$. □

Remark 3.6. We avoided introducing the cross-section $\Gamma(\theta_\varepsilon) := \frac{d\rho}{d\theta_\varepsilon}$ of the problem because the map $\rho \rightarrow \theta_\varepsilon(\rho)$ is not monotonic in general.

Indeed if ϕ is bounded and ε sufficiently small, $\frac{1}{2}v^2 > \varepsilon^\alpha \phi(0)$ so that $\theta = 0$ for $\rho = 0$ and $\rho = \pm 1$. As a consequence, $\Gamma(\theta_\varepsilon)$ is neither single valued nor bounded.

3.2. Proof of Theorem 2.2. The general structure of the proof follows the lines of [DP] where an analogous result has been proven when the magnetic field is zero.

Proposition 3.7. Let f_ε be defined in (2.5). Then, for any $T > 0$,

$$(3.14) \quad \lim_{\varepsilon \rightarrow 0} \|f_\varepsilon - h_{\varepsilon,\gamma}\|_{L^\infty([0,T]; L^1(\mathbb{R}^2 \times S_1))} = 0$$

where $h_{\varepsilon,\gamma}$ is the unique solution of the truncated linear Boltzmann equation with magnetic field

$$(3.15) \quad \begin{cases} (\partial_t + v \cdot \nabla_x + B v^\perp \cdot \nabla_v) h_{\varepsilon,\gamma}(t, x, v) = \tilde{L} h_{\varepsilon,\gamma}(t, x, v) \\ h_{\varepsilon,\gamma}(x, v, 0) = f_0(x, v) \end{cases},$$

with

$$\tilde{L}f(v) = \mu \int_{-\pi}^{\pi} \Gamma_{\varepsilon,\gamma}^{(B)}(\theta) \{f(\mathcal{R}(\theta)v) - f(v)\} d\theta.$$

and $\Gamma_{\varepsilon,\gamma}^{(B)}$ is the differential cross section associated to the unrescaled potential ψ_ε with magnetic field.

The proof of Proposition 3.7 is in Section 5.

This allows to reduce the problem of the transition from the solution of the truncated linear Boltzmann equation to the solution of the untruncated linear Boltzmann equation to a partial differential equation problem. Indeed, as in [DP], we can prove the following

Proposition 3.8. *Let $h_{\varepsilon,\gamma}$ solution of (3.15). Then, for any $T > 0$,*

$$(3.16) \quad h_{\varepsilon,\gamma} \rightarrow f \quad \text{in} \quad C([0, T]; \mathcal{D}')$$

where f is the unique solution of (2.6).

Proof. In Appendix B, Proposition B.2, the cross section $\Gamma_{\varepsilon,\gamma}^{(B)}(\theta)$ is shown to be bounded by $C\theta^{-1-1/s}$ and to converge to $\Gamma(\theta)$ almost everywhere as $\varepsilon \rightarrow 0$, where $\Gamma(\theta)$ is the cross section associated to the truly long range potential $\Psi(r) = r^{-s}$ without magnetic field. Therefore, the proof of Proposition 3.8 is exactly the same as the one of Proposition A.2 in [DP]. \square

4. PROOF OF PROPOSITION 3.1

In this Section we prove the asymptotic equivalence of f_ε , defined by (2.2), and h_ε , solution of the linear Boltzmann equation (3.1), that we recall here for the sake of clarity

$$(4.1) \quad (\partial_t + v \cdot \nabla_x + B v^\perp \cdot \nabla_v) h_\varepsilon(x, v, t) = L_\varepsilon h_\varepsilon(x, v, t),$$

where

$$(4.2) \quad L_\varepsilon h_\varepsilon(v) = \mu\varepsilon^{-2\alpha} \int_{-1}^1 d\rho \{h_\varepsilon(v') - h_\varepsilon(v)\}.$$

This allows to reduce the problem to the analysis of a Markov process which is an easier task. Indeed, the series expansion defining h_ε (obtained perturbing around the loss term) reads as

$$(4.3) \quad h_\varepsilon(x, v, t) = e^{-2\varepsilon^{-2\alpha}\mu t} \sum_{Q \geq 0} \mu_\varepsilon^Q \int_0^t dt_Q \dots \int_0^{t_2} dt_1 \int_{-\varepsilon}^\varepsilon d\rho_1 \dots \int_{-\varepsilon}^\varepsilon d\rho_Q f_0(\gamma^{-t}(x, v)).$$

Here $\gamma^{-t}(x, v) = (\xi_\varepsilon(-t), \eta_\varepsilon(-t))$ where η_ε is an autonomous jump process and ξ_ε is an additive functional of η_ε . Equation (4.3) is an evolution equation for the probability density associated to a particle performing random jumps in the velocity space at random Markov times.

We start the proof by looking at the microscopic solution f_ε defined by (1.5). For $(x, v) \in \mathbb{R}^2 \times \mathbb{R}^2$, $t > 0$, we have

$$(4.4) \quad f_\varepsilon(x, v, t) = e^{-\mu\varepsilon|\mathcal{B}(x,t)|} \sum_{N \geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{\mathcal{B}(x,t)^N} d\mathbf{c}_N f_0(T_{\mathbf{c}_N}^{-t}(x, v)),$$

where $T_{\mathbf{c}_N}^t(x, v)$ is the Hamiltonian flow with initial datum (x, v) and $\mathcal{B}(x, t)$ is the disk centered in x with radius t .

Given the configuration of obstacles $\mathbf{c}_N = c_1 \dots c_N$, we shall say that c_i is *internal* if it influences the motion up to the time t , i.e.

$$(4.5) \quad \inf_{0 \leq s \leq t} |x_\varepsilon(-s) - c_i| < \varepsilon,$$

while we shall call c_i *external* if

$$(4.6) \quad \inf_{0 \leq s \leq t} |x_\varepsilon(-s) - c_i| \geq \varepsilon.$$

Here $(x_\varepsilon(-s), v_\varepsilon(-s)) = T_{\mathbf{c}_N}^{-s}(x, v)$, $s \in [0, t]$.

We can perform the integration over the external obstacles and we get

$$(4.7) \quad f_\varepsilon(x, v, t) = \sum_{Q \geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{\mathcal{B}(x,t)^Q} d\mathbf{b}_Q e^{-\mu_\varepsilon|\mathcal{T}(\mathbf{b}_Q)|} \chi(\{\mathbf{b}_Q \text{ internal}\}) f_0(T_{\mathbf{b}_Q}^{-t}(x, v)),$$

where $\chi(E)$ is the characteristic function of the event E and $\mathcal{T}(\mathbf{b}_Q)$ is the tube

$$(4.8) \quad \mathcal{T}(\mathbf{b}_Q) = \{y \in \mathcal{B}(x, t) \text{ s.t. } \exists s \in (0, t) \text{ s.t. } |y - x_\varepsilon(-s)| < \varepsilon\}.$$

Note that in the previous integration we are not considering possible overlappings of obstacles. This is legitimate because we shall see that this event is negligible as ε tends to 0.

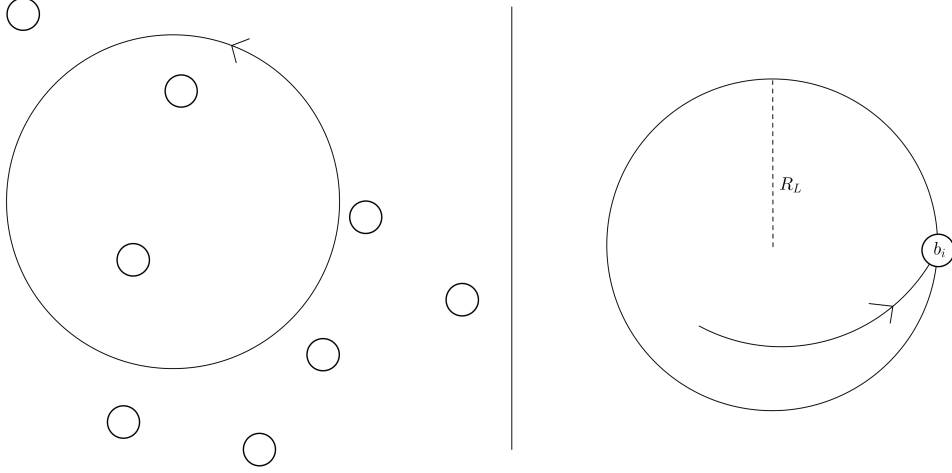


FIGURE 2. On the left a cyclotron orbit completed without suffering collisions is represented. On the right there is a repeated collision with the same scatterer.

Furthermore, let us restrict to the configurations such that the light particle's trajectory does not start from inside an obstacle and does not end inside an obstacle: in formula

$$(4.9) \quad \chi_1(\mathbf{b}_Q) = \chi\{\mathbf{b}_Q \text{ s.t. } b_i \notin \mathcal{B}(x, \varepsilon) \text{ and } b_i \notin \mathcal{B}(x_\varepsilon(-t), \varepsilon) \text{ for all } i = 1, \dots, Q\}.$$

As for the overlappings, this choice is not really restrictive because the contribution related to $1 - \chi_1$ is going to vanish in the limit, as we shall see. Moreover, we will now list other events that will turn out to be negligible as ε approaches 0.

i) **Complete cyclotronic orbit.**

A first cyclotron orbit is completed without suffering any collisions and a repeated collision occurs with the same scatterer without any collision in the meantime.

We set

$$(4.10) \quad \chi_{circ}(\mathbf{b}_Q) = \chi(\{\mathbf{b}_Q \text{ s.t. } i \text{ is realized}\}).$$

For a pictorial representation of the event i) see Figure 2. We now define

$$(4.11) \quad \check{f}_\varepsilon(x, v, t) = \sum_{Q \geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{\mathcal{B}(x, t)^Q} d\mathbf{b}_Q e^{-\mu_\varepsilon |\mathcal{T}(\mathbf{b}_Q)|} \chi(\{\mathbf{b}_Q \text{ internal}\}) \\ (1 - \chi_{circ}(\mathbf{b}_Q)) \chi_1(\mathbf{b}_Q) f_0(T_{\mathbf{b}_Q}^{-t}(x, v)).$$

Note that $f_\varepsilon \geq \check{f}_\varepsilon$. For $t < T_L$ one expects that the approximation with the dynamics of the test particle in absence of the external field is true. The unexpected fact is that even for $t \gtrsim O(T_L)$ this still holds because (4.10) tends to 0 as $\varepsilon \rightarrow 0$. Hence, for a given configuration \mathbf{b}_Q such that $\chi_1[1 - \chi_{circ}](\mathbf{b}_Q) = 1$, we have that the measure of the tube can be estimated by

$$(4.12) \quad |\mathcal{T}(\mathbf{b}_Q)| \leq 2\varepsilon t.$$

At this point we define

$$(4.13) \quad \tilde{f}_\varepsilon(x, v, t) = e^{-2\mu_\varepsilon^{-2\alpha} t} \sum_{Q \geq 0} \frac{\mu_\varepsilon^Q}{Q!} \int_{\mathcal{B}(x, t)^Q} d\mathbf{b}_Q \chi(\{\mathbf{b}_Q \text{ internal}\}) \\ (1 - \chi_{circ}(\mathbf{b}_Q)) \chi_1(\mathbf{b}_Q) f_0(T_{\mathbf{b}_Q}^{-t}(x, v)).$$

Thanks to (4.12) we get

$$(4.14) \quad f_\varepsilon \geq \check{f}_\varepsilon \geq \tilde{f}_\varepsilon.$$

According to a classical argument introduced in [G] (see also [DP, DR, BNP]), we now want to remove from \tilde{f}_ε all the events that prevent the light particle's trajectory to be the Markov process described by h_ε .

For any fixed initial condition (x, v) we order the obstacles b_1, \dots, b_N according to the scattering sequence. Let ρ_i and t_i be the impact parameter and the backwards entrance time of the light particle in the protection disk around b_i , namely $\mathcal{B}(b_i, \varepsilon)$. Then we perform the following change of variables

$$(4.15) \quad b_1, \dots, b_N \rightarrow \rho_1, t_1, \dots, \rho_N, t_N$$

with

$$0 \leq t_N < t_{N-1} < \dots < t_1 \leq t.$$

Conversely, fixed the impact parameters $\{\rho_i\}$ and the hitting times $\{t_i\}$ we construct the centers of the obstacles $b_i = b(\rho_i, t_i)$ and a trajectory $\bar{\gamma}^{-s}(x, v) := (\bar{\xi}_\varepsilon(-s), \bar{\eta}_\varepsilon(-s))$, $s \in [0, t]$ inductively.

Suppose that we are able to define the obstacles b_1, \dots, b_{i-1} and a trajectory $\bar{\gamma}^{-s}(x, v) := (\bar{\xi}_\varepsilon(-s), \bar{\eta}_\varepsilon(-s))$ up to the time $s = t_{i-1}$. We then define the trajectory between times t_{i-1} and t_i as that of the evolution of a particle moving under the action of the Lorentz force and of the potential $\varepsilon^\alpha \phi(\varepsilon^{-1}|\cdot - b_{i-1}|)$ with initial datum $(\bar{\xi}_\varepsilon(-t_{i-1}), \bar{\eta}_\varepsilon(-t_{i-1}))$. Then b_i is defined to be the only point at distance ε of $\bar{\xi}_\varepsilon(-t_i)$ and algebraic distance ρ_i from the straight line which is tangent to the trajectory at the point $\bar{\xi}_\varepsilon(-t_i)$.

However, $\bar{\gamma}^{-s}(x, v) = (x_\varepsilon(-s), v_\varepsilon(-s))$ (therefore the mapping (4.15) is one-to-one) only outside the following pathological situations (relative to the backward trajectory).

ii) **Overlapping.**

If b_i and b_j are both internal then $\mathcal{B}(b_i, \varepsilon) \cap \mathcal{B}(b_j, \varepsilon) \neq \emptyset$.

iii) **Recollisions.**

There exists b_i such that for $s \in (t_{j+1}, t_j)$, $j > i$, $\xi_\varepsilon(-s) \in \mathcal{B}(b_i, \varepsilon)$.

iv) **Interferences.**

There exists b_i such that $\xi_\varepsilon(-s) \in \mathcal{B}(b_j, \varepsilon)$ for $s \in (t_{i+1}, t_i)$, $j > i$.

We simply skip such events by setting

$$\chi_{ov} = \chi(\{\mathbf{b}_Q \text{ s.t. ii) is realized}\}),$$

$$\chi_{rec} = \chi(\{\mathbf{b}_Q \text{ s.t. iii) is realized}\}),$$

$$\chi_{int} = \chi(\{\mathbf{b}_Q \text{ s.t. iv) is realized}\}),$$

and defining

$$(4.16) \quad \bar{f}_\varepsilon(x, v, t) = e^{-2\varepsilon^{-2\alpha}\mu t} \sum_{Q \geq 0} \mu_\varepsilon^Q \int_0^t dt_1 \dots \int_0^{t_{Q-1}} dt_Q \int_{-\varepsilon}^\varepsilon d\rho_1 \dots \int_{-\varepsilon}^\varepsilon d\rho_Q \chi_1(1 - \chi_{circ})(1 - \chi_{ov})(1 - \chi_{rec})(1 - \chi_{int}) f_0(\bar{\gamma}^{-t}(x, v)).$$

Note that

$$(4.17) \quad \bar{f}_\varepsilon \leq \tilde{f}_\varepsilon \leq \check{f}_\varepsilon \leq f_\varepsilon.$$

Note also that in (4.16) we have used the change of variables (4.15) for which, outside the pathological sets i), ii), iii), iv) $\bar{\gamma}^{-t}(x, v) = (x_\varepsilon(-t), v_\varepsilon(-t))$.

Next we remove $\chi_1(1 - \chi_{circ})(1 - \chi_{ov})(1 - \chi_{rec})(1 - \chi_{int})$ by setting

$$(4.18) \quad \bar{h}_\varepsilon(x, v, t) = e^{-2\varepsilon^{-2\alpha}\mu t} \sum_{Q \geq 0} \mu_\varepsilon^Q \int_0^t dt_1 \dots \int_0^{t_{Q-1}} dt_Q \int_{-\varepsilon}^\varepsilon d\rho_1 \dots \int_{-\varepsilon}^\varepsilon d\rho_Q f_0(\bar{\gamma}^{-t}(x, v)).$$

We observe that

$$(4.19) \quad 1 - \chi_1(1 - \chi_{ov})(1 - \chi_{cir})(1 - \chi_{rec})(1 - \chi_{int}) \leq (1 - \chi_1) + \chi_{ov} + \chi_{cir} + \chi_{rec} + \chi_{int}.$$

Then by (4.16) and (4.18) we obtain

$$|\bar{h}_\varepsilon(t) - \bar{f}_\varepsilon(t)| \leq \varphi_1(\varepsilon, t)$$

with

$$(4.20) \quad \varphi_1(\varepsilon, t) = \|f_0\|_\infty e^{-2\varepsilon^{-2\alpha}\mu t} \sum_{Q \geq 0} \mu_\varepsilon^Q \int_0^t dt_1 \dots \int_0^{t_{Q-1}} dt_Q \int_{-\varepsilon}^\varepsilon d\rho_1 \dots \int_{-\varepsilon}^\varepsilon d\rho_Q [(1 - \chi_1) + \chi_{ov} + \chi_{cir} + \chi_{rec} + \chi_{int}].$$

We can prove that φ_1 is negligible in the limit. The precise statement follows.

Proposition 4.1. *Let $\varphi_1(\varepsilon, t)$ be defined as in (4.20). For any $t \in [0, T]$*

$$\|\varphi_1(\varepsilon, t)\|_{L^1} \rightarrow 0$$

as $\varepsilon \rightarrow 0$.

Proof. See Section 4.1. □

To complete the proof of Theorem 2.1 we still need to show the asymptotic equivalence of \bar{h}_ε and h_ε . Notice that h_ε is given by (4.3) where the trajectory $\gamma^{-t}(x, v) = (\xi_\varepsilon(-t), \eta_\varepsilon(-t))$ is a jump process in the velocity space, i.e. the changes of velocity are instantaneous. We compare the trajectory $\bar{\gamma}^{-t}(x, v) = (\bar{\xi}_\varepsilon(-t), \bar{\eta}_\varepsilon(-t))$ with $\gamma^{-t}(x, v)$: being t_1, \dots, t_Q the sequence of impact times and $\tau \leq C\varepsilon$ the collision time, the spacial coordinates can differ only inside the interaction disk, while the velocities can differ only if $t \in (t_1, t_1 + \tau)$.

In formulae we have

$$(4.21) \quad \begin{aligned} |\xi_\varepsilon(-t) - \bar{\xi}_\varepsilon(-t)| &\leq C_1 Q \varepsilon \\ |\eta_\varepsilon(-t) - \bar{\eta}_\varepsilon(-t)| &\leq C_2 \varepsilon^\alpha \chi(t - t_1 \leq C\varepsilon). \end{aligned}$$

By exploiting the regularity of the initial condition f_0 we get

$$(4.22) \quad |f_0(\xi_\varepsilon(-t), \eta_\varepsilon(-t)) - f_0(\bar{\xi}_\varepsilon(-t), \bar{\eta}_\varepsilon(-t))| \leq C'[Q\varepsilon + \varepsilon^\alpha \chi(t - t_1 \leq C\varepsilon)]$$

which implies

$$(4.23) \quad \begin{aligned} |h_\varepsilon(x, v, t) - \bar{h}_\varepsilon(x, v, t)| &\leq C' e^{-2\mu t \varepsilon^{-2\alpha}} \sum_{Q \geq 0} \mu_\varepsilon^Q \int_0^t dt_1 \dots \int_0^{t_{Q-1}} dt_Q \\ &\quad \times \int_{-\varepsilon}^\varepsilon d\rho_1 \dots \int_{-\varepsilon}^\varepsilon d\rho_Q [Q\varepsilon + \varepsilon^\alpha \chi(t - t_1 \leq C\varepsilon)] \\ &\leq C(t\varepsilon^{1-2\alpha} + \varepsilon^{1+\alpha}). \end{aligned}$$

Hence we obtained $\lim_{\varepsilon \rightarrow 0} \|h_\varepsilon - \bar{h}_\varepsilon\|_{L^\infty([0, T] \times \mathbb{R}^2 \times S_1)} = 0$. We observe that the monotonicity argument behind this strategy, see equation (4.17), the positivity of the solution h_ε of the Boltzmann equation and the conservation of mass imply that $\bar{f}_\varepsilon, \bar{h}_\varepsilon$ and h_ε have the same asymptotic behavior in $L^\infty([0, T]; L^1(\mathbb{R}^2 \times S_1))$ when $\varepsilon \rightarrow 0$.

4.1. Control of the pathological sets: proof of Proposition 4.1. In this section we prove Proposition 4.1. This makes rigorous the claim of the heuristic argument presented in the paper's introduction. For any measurable function u of the backward Markov process $(\xi_\varepsilon, \eta_\varepsilon)$ we set

$$\begin{aligned} \mathbb{E}_{x,v}[u] &= e^{-2\mu_\varepsilon \varepsilon t} \sum_{Q \geq 0} \mu_\varepsilon^Q \int_0^t dt_1 \dots \int_0^{t_{Q-1}} dt_Q \\ &\quad \int_{-\varepsilon}^\varepsilon d\rho_1 \dots \int_{-\varepsilon}^\varepsilon d\rho_Q u(\xi_\varepsilon, \eta_\varepsilon). \end{aligned}$$

Recalling (4.19) we have

$$\varphi_1(\varepsilon, t) \leq \|f_0\|_\infty \mathbb{E}_{x,v}[(1 - \chi_1) + \chi_{ov} + \chi_{cir} + \chi_{rec} + \chi_{int}].$$

We can skip the estimates of the first two contributions, i.e. $\mathbb{E}_{x,v}[(1 - \chi_1)]$ and $\mathbb{E}_{x,v}[\chi_{ov}]$, since the presence of the external field does not affect the classical arguments which can be found in [BNP, DR, DP]. However, the presence of the magnetic field and consequently the circular motion of the test particle strongly affects the explicit estimates of the pathological events ii), iii). Therefore, we need a detailed analysis for χ_{cir} , χ_{rec} and χ_{int} .

For what concerns the pathological event due to a recollision with the same scatterer (see Figure 3), we observe that $\chi_{cir} = 1$ if there exists an entrance time t_i such that $|t_i - t_{i+1}| \geq T_L - \tau \geq T_L - C\varepsilon$ for some $i = 0, \dots, Q - 1$. Moreover, $\chi_{circ} = 1$ also when a test particle performs an entire Larmor orbit without colliding with any obstacles. As explained in the introduction, the probability of this event is bounded from above by $C \exp(-\frac{2\pi\mu}{B} \varepsilon^{-2\alpha}) := c_\alpha(\varepsilon)$. Therefore, it results

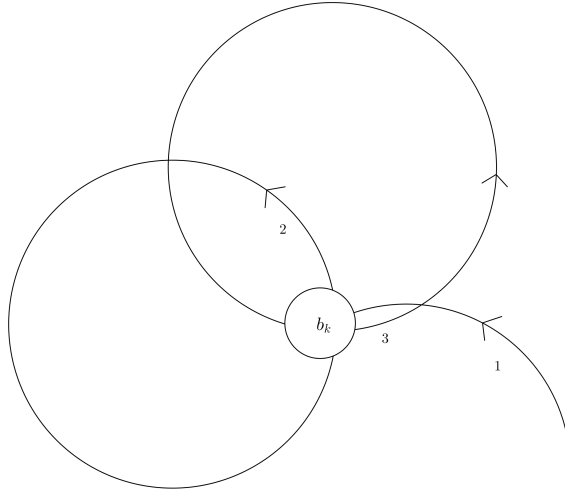


FIGURE 3. Recollision with the same scatter.

$$\begin{aligned}
 \mathbb{E}_{x,v}[\chi_{circ}] &\leq c_\alpha(\varepsilon) + e^{-2\mu_\varepsilon t} \sum_{Q \geq 1} \mu_\varepsilon^Q \int_0^t dt_1 \cdots \int_0^{t_{Q-1}} dt_Q \int_{-\varepsilon}^\varepsilon d\rho_1 \cdots \int_{-\varepsilon}^\varepsilon d\rho_Q \sum_{i=1}^Q \chi(t_i < t_{i-1} - T_L + C\varepsilon) \\
 (4.24) \quad &\leq c_\alpha(\varepsilon) + e^{-2t\mu_\varepsilon} \sum_{Q \geq 1} (2\mu_\varepsilon \varepsilon)^Q \int_0^t dt_1 \cdots \int_0^{t_{Q-1}} dt_Q \sum_{i=1}^Q \chi(t_i < t_{i-1} - T_L + C\varepsilon)
 \end{aligned}$$

where $\mu_\varepsilon = \mu \varepsilon^{-1-2\alpha}$ with $\alpha \in (0, 1/2)$ and $t_0 = t$.

We set

$$\begin{aligned}
 I_i &:= \int_0^t dt_1 \cdots \int_0^{t_{Q-1}} dt_Q \chi(t_i < t_{i-1} - T_L + C\varepsilon) \\
 &= \int_0^t dt_1 \cdots \int_0^{t_{i-1}} dt_i \chi(t_i < t_{i-1} - T_L + C\varepsilon) \frac{t_i^{Q-i}}{(Q-i)!} \\
 &= \int_0^t dt_1 \cdots \int_0^{t_{i-2}} dt_{i-1} \frac{(t_{i-1} - T_L + C\varepsilon)^{Q-(i-1)}}{(Q-(i-1))!}
 \end{aligned}$$

then

$$(4.25) \quad \mathbb{E}_{x,v}[\chi_{circ}] \leq c(\varepsilon) + e^{-2t\mu_\varepsilon} \sum_{Q \geq 1} (2\mu_\varepsilon \varepsilon)^Q \sum_{i=1}^Q I_i.$$

Note that

$$\begin{aligned}
 I_{i+1} &= \int_0^t dt_1 \cdots \int_0^{t_{i-1}} dt_i \frac{(t_i - T_L + C\varepsilon)^{Q-i}}{(Q-i)!} \\
 &= \int_0^t dt_1 \cdots \int_0^{t_{i-2}} dt_{i-1} \frac{(t_{i-1} - T_L + C\varepsilon)^{Q-(i-1)} - (-T_L + C\varepsilon)^{Q-(i-1)}}{(Q-(i-1))!} \\
 &= I_i - \frac{(-T_L + C\varepsilon)^{Q-i+1}}{(Q-i+1)!} \frac{t^{i-1}}{(i-1)!} \\
 &= I_i - f_{i-1}
 \end{aligned}$$

where $f_i := \frac{(-T_L + C\varepsilon)^{Q-i}}{(Q-i)!} \frac{t^i}{(i)!}$. We now look at

$$\begin{aligned}
\sum_{i=1}^Q I_i &= QI_1 - \sum_{j=0}^{Q-2} (Q-1-j)f_j \\
&= \frac{1}{(Q-1)!} \left[(t - T_L + C\varepsilon)^Q - \sum_{j=0}^{Q-2} (Q-1)!(Q-j) \frac{(-T_L + C\varepsilon)^{Q-j}}{(Q-j)!} \frac{t^j}{j!} + \sum_{j=0}^{Q-2} (Q-1)! \frac{(-T_L + C\varepsilon)^{Q-j}}{(Q-j)!} \frac{t^j}{j!} \right] \\
&\leq \frac{1}{(Q-1)!} \left[2(t - T_L + C\varepsilon)^Q + (T_L - C\varepsilon)(t - T_L + C\varepsilon)^{Q-1} \right].
\end{aligned}
\tag{4.26}$$

Finally we got

$$\begin{aligned}
\mathbb{E}_{x,v}[\chi_{circ}] &\leq c_\alpha(\varepsilon) + e^{-2t\mu_\varepsilon\varepsilon} \sum_{Q \geq 1} (2\mu_\varepsilon\varepsilon)^Q \left[\frac{2(t - T_L + C\varepsilon)^Q}{(Q-1)!} + \frac{(T_L - C\varepsilon)(t - T_L + C\varepsilon)^{Q-1}}{(Q-1)!} \right] \\
&= c_\alpha(\varepsilon) + 2e^{-2t\mu_\varepsilon\varepsilon} (2\mu_\varepsilon\varepsilon)(t - T_L + C\varepsilon) \sum_{Q \geq 1} (2\mu_\varepsilon\varepsilon)^{Q-1} \frac{(t - T_L + C\varepsilon)^{Q-1}}{(Q-1)!} \\
&\quad + e^{-2t\mu_\varepsilon\varepsilon} (T_L - C\varepsilon)(2\mu_\varepsilon\varepsilon) \sum_{Q \geq 1} (2\mu_\varepsilon\varepsilon)^{Q-1} \frac{(t - T_L + C\varepsilon)^{Q-1}}{(Q-1)!} \\
&= c_\alpha(\varepsilon) + 2e^{-2(T_L - C\varepsilon)\mu_\varepsilon\varepsilon^{-2\alpha}} (2\mu_\varepsilon\varepsilon^{-2\alpha})(t - T_L + C\varepsilon) + e^{-2(T_L - C\varepsilon)\mu_\varepsilon\varepsilon^{-2\alpha}} (T_L - C\varepsilon)(2\mu_\varepsilon\varepsilon^{-2\alpha})
\end{aligned}
\tag{4.27}$$

for $\alpha > 0$ and ε sufficiently small. Hence, $\mathbb{E}_{x,v}[\chi_{circ}]$ vanishes as $\varepsilon \rightarrow 0$.

We now consider a generalization of χ_{cirl} : let be $\chi_{arc}^{(\nu)}$ the characteristic function of the event such that the light particle does not hit any obstacles in a time interval equal to $T_L \varepsilon^\nu$, $0 < \nu < 1$. More precisely $\chi_{arc}^{(\nu)} = 1$ if there exists an entrance time t_i such that $|t_i - t_{i+1}| \geq T_L \varepsilon^\nu - \tau \geq T_L \varepsilon^\nu - C\varepsilon$ for some $i = 0, \dots, Q-1$. The same computations as for $\mathbb{E}_{x,v}[\chi_{cirl}]$ show that $\mathbb{E}_{x,v}[\chi_{arc}^{(\nu)}]$ vanishes as $\varepsilon \rightarrow 0$ when $\nu < 2\alpha$. In other words this shows that the motion of the light particle outside the obstacles covers arcs of circle and corresponding angles of order at most $O(\varepsilon^\nu)$.

Next we pass to the control of the recollision event. We observe that

$$\chi_{rec} = (1 - \chi_{arc}^{(\nu)})\chi_{rec} + \chi_{rec} \chi_{arc}^{(\nu)} \leq (1 - \chi_{arc}^{(\nu)})\chi_{rec} + \chi_{arc}^{(\nu)}
\tag{4.28}$$

and this implies

$$\mathbb{E}_{x,v}[\chi_{rec}] \leq \mathbb{E}_{x,v}[(1 - \chi_{arc}^{(\nu)})\chi_{rec}] + \mathbb{E}_{x,v}[\chi_{arc}^{(\nu)}],$$

but $\mathbb{E}_{x,v}[\chi_{arc}^{(\nu)}]$ is vanishing in the limit $\varepsilon \rightarrow 0$ as we have seen before. Therefore, we can focus on $\mathbb{E}_{x,v}[(1 - \chi_{arc}^{(\nu)})\chi_{rec}]$. Let t_i the first time the light particle hits the i -th scatterer b_i , v_i^- the incoming velocity, v_i^+ the outgoing velocity (with respect to the backwards trajectory) and t_i^+ the exit time. Moreover, we fix the axis in such a way that v_i^+ is parallel to the x axis. We have

$$\chi_{rec}(1 - \chi_{arc}^{(\nu)}) \leq (1 - \chi_{arc}^{(\nu)}) \sum_{i=1}^Q \sum_{j>1} \chi_{rec}^{i,j},
\tag{4.29}$$

where $\chi_{rec}^{i,j} = 1$ if and only if b_i (constructed via the sequence $t_1, \rho_1, \dots, t_i, \rho_i$) is recollided in the time interval (t_j, t_{j-1}) . Note indeed that a recollision can occur only if the rotation angle $|\sum_{h=i+1}^{j-1} (\theta_h + \varphi_h)| > \pi$ where φ_h is the angle covered outside the obstacles in the time interval (t_{h+1}, t_h^+) , being θ_h the h -th scattering angle. The constraint $(1 - \chi_{arc}^{(\nu)})$ implies that $|\varphi_h| \leq C'\varepsilon^\nu$.

Hence, since $|\theta_h + \varphi_h| \leq C\varepsilon^\alpha + C'\varepsilon^\nu \leq C''\varepsilon^{\nu/2}$, in order to have a recollision there must be an intermediate velocity v_k , $k = i+1, \dots, j-1$ such that

$$|v_k^+ \cdot v_j^+| \leq C\varepsilon^{\nu/2},
\tag{4.30}$$

namely v_k^+ is almost orthogonal to v_j^+ (see Figure 4).

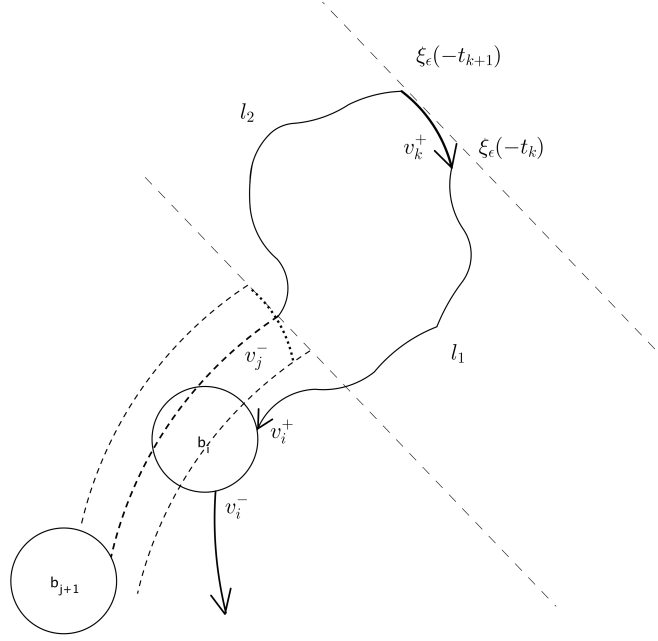


FIGURE 4. Backward recollision.

Then

$$(4.31) \quad \chi_{rec}(1 - \chi_{arc}^{(\nu)}) \leq (1 - \chi_{arc}^{(\nu)}) \sum_{i=1}^Q \sum_{j=i+2}^Q \sum_{k=i+1}^{j-1} \chi_{rec}^{i,j,k},$$

where $\chi_{rec}^{i,j,k} = 1$ if and only if $\chi_{rec}^{i,j} = 1$ and (4.30) is fulfilled. Following [BNP], we fix all the parameters $\rho_1, \dots, \rho_Q, t_1, \dots, t_Q$ but t_{k+1} . The two branches of the trajectory l_1, l_2 are rigid so that, when a recollision occurs, the integration domain with respect to t_{k+1} is restricted to a time interval bounded by

$$\frac{2\varepsilon}{\cos C\varepsilon^{\nu/2}} \leq 4\varepsilon.$$

Performing all the other integrations and summing over i, j, k we obtain

$$(4.32) \quad \begin{aligned} & \mathbb{E}_{x,v} \left[(1 - \chi_{arc}) \sum_{i=1}^Q \sum_{j=1}^Q \sum_{k=i+1}^{j-1} \chi_{rec}^{i,j,k} \right] \\ & \leq C\varepsilon e^{-2\mu\varepsilon^{-2\alpha}t} \sum_{Q \geq 3} (Q-1)(Q-2)(Q-3) \frac{(2\mu\varepsilon^{-2\alpha})^Q}{(Q-1)!} t^{Q-1} \\ & \leq C't^3\varepsilon^{1-8\alpha}, \end{aligned}$$

which tends to 0 as ε goes to 0 for $\alpha < 1/8$.

Following the strategy used in [BNP], since a backward interference is a forward recollision, the estimate for the interference event can be handled by using the Liouville Theorem.

5. PROOF OF PROPOSITION 3.7

Our aim is to prove the asymptotic equivalence of f_ε , defined by (2.5), and $h_{\varepsilon,\gamma}$ solution of the linear Boltzmann equation (3.15), namely

$$\begin{cases} (\partial_t + v \cdot \nabla_x + B v^\perp \cdot \nabla_v) h_{\varepsilon,\gamma}(t, x, v) = \tilde{L} h_{\varepsilon,\gamma}(t, x, v) \\ h_{\varepsilon,\gamma}(x, v, 0) = f_0(x, v) \end{cases},$$

which reads

$$(5.1) \quad h_{\varepsilon,\gamma}(x, v, t) = e^{-2\mu\varepsilon^{\gamma-1}t} \sum_{Q \geq 0} \mu_\varepsilon^Q \int_0^t dt_Q \dots \int_0^{t_2} dt_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \dots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q f_0(\bar{\xi}_\varepsilon(-t), \bar{\eta}_\varepsilon(-t)).$$

We recall that we can expand f_ε as follows:

$$(5.2) \quad f_\varepsilon(x, v, t) = e^{-\mu_\varepsilon |\mathcal{B}(x, t)|} \sum_{N \geq 0} \frac{\mu_\varepsilon^N}{N!} \int_{\mathcal{B}(x, t)^N} d\mathbf{c}_N f_0(T_{\mathbf{c}_N}^{-t}(x, v)),$$

where $T_{\mathbf{c}_N}^t(x, v)$ is the Hamiltonian flow with initial datum (x, v) and $\mathcal{B}(x, t)$ is the disk of center x and radius t . We observe that the proof follows the same strategy of Proposition 3.1 (see also Section 3 in [DP]). As before the hardest part is the estimate of the non-Markovian contribution which is summarized in the following proposition.

Proposition 5.1. *Let $\varphi_1(\varepsilon, t)$ be defined as in (4.20) with the only difference that the radius of the obstacles is now ε^γ instead of ε and the collision time τ is bounded by $C\varepsilon^\gamma$ instead of $C\varepsilon$. Then for any $t \in [0, T]$*

$$\|\varphi_1(\varepsilon, t)\|_{L^1} \rightarrow 0$$

as $\varepsilon \rightarrow 0$.

Proof. Also in this case we can skip the estimates of the contributions $\mathbb{E}_{x,v}[(1 - \chi_1)]$ and $\mathbb{E}_{x,v}[\chi_{ov}]$ since the presence of the external field does not affect the bounds in [DP]. Moreover, as in proposition 4.1, if we know that $\mathbb{E}_{x,v}[\chi_{rec}]$ is negligible, then the Liouville theorem guarantees that also $\mathbb{E}_{x,v}[\chi_{int}]$ can be disregarded in the limit. Hence, it suffices to focus on $\mathbb{E}_{x,v}[\chi_{circ}]$ and $\mathbb{E}_{x,v}[\chi_{rec}]$. So we look at

$$(5.3) \quad \mathbb{E}_{x,v}[\chi_{circ}] \leq c_\gamma(\varepsilon) + e^{-2t\mu_\varepsilon\varepsilon^\gamma} \sum_{Q \geq 1} (2\mu_\varepsilon\varepsilon^\gamma)^Q \int_0^t dt_1 \cdots \int_0^{t_{Q-1}} dt_Q \sum_{i=0}^{Q-1} \chi(t_{i+1} < t_i - T_L + C\varepsilon^\gamma)$$

where $c_\gamma(\varepsilon) = C \exp(-\frac{2\pi\mu}{B}\varepsilon^{\gamma-1})$, $\mu_\varepsilon = \mu\varepsilon^{-1}$ and $\gamma \in (0, 1)$. Following the same strategy as in proposition 4.1, we obtain

$$(5.4) \quad \begin{aligned} \mathbb{E}_{x,v}[\chi_{circ}] &\leq c_\gamma(\varepsilon) + e^{-2t\mu_\varepsilon\varepsilon^\gamma} \sum_{Q \geq 1} (2\mu_\varepsilon\varepsilon^\gamma)^Q \left[\frac{2(t - T_L + C\varepsilon^\gamma)^Q}{(Q-1)!} + \frac{(T_L - C\varepsilon^\gamma)(t - T_L + C\varepsilon^\gamma)^{Q-1}}{(Q-1)!} \right] \\ &= c_\gamma(\varepsilon) + 2e^{-2(T_L - C\varepsilon^\gamma)\mu\varepsilon^{\gamma-1}} (2\mu\varepsilon^{\gamma-1})(t - T_L + C\varepsilon^\gamma) + e^{-2(T_L - C\varepsilon^\gamma)\mu\varepsilon^{\gamma-1}} (2\mu\varepsilon^{\gamma-1})(T_L - C\varepsilon^\gamma) \end{aligned}$$

which vanishes as $\varepsilon \rightarrow 0$ and $\gamma \in (0, 1)$.

To control the recollision event we can follow the strategy used in Section 4.1 and in [DP], Proposition 3.1. More precisely, as in Section 4.1, we introduce $\chi_{arc}^{(M)}$ such that $\chi_{arc}^{(M)} = 1$ if there exists an entrance time t_i such that $|t_i - t_{i+1}| \geq T_L/M - \tau \geq T_L/M - C\varepsilon^\gamma$ for some $i = 0, \dots, Q-1$ where M is a finite constant and $M > 1$. One can easily see that $\mathbb{E}_{x,v}[\chi_{arc}^{(M)}]$ vanishes as $\varepsilon \rightarrow 0$ when $\gamma < 1$. Furthermore,

$$\mathbb{E}_{x,v}[\chi_{rec}] \leq \mathbb{E}_{x,v}[(1 - \chi_{arc}^{(M)})\chi_{rec}] + \mathbb{E}_{x,v}[\chi_{arc}^{(M)}]$$

but $\mathbb{E}_{x,v}[\chi_{arc}^{(M)}]$ is vanishing in the limit $\varepsilon \rightarrow 0$ as we have seen before. Therefore, we can focus on $\mathbb{E}_{x,v}[(1 - \chi_{arc}^{(M)})\chi_{rec}]$. We now distinguish the collisions as

$$(5.5) \quad \begin{aligned} (1 - \chi_{arc}^{(M)})\chi_{rec} &\leq (1 - \chi_{arc}^{(M)}) \sum_{i=1}^Q \sum_{j>1} \chi_{rec}^{i,j} \chi(\sin \alpha_{jk} \leq \frac{\varepsilon^\delta}{4}, \forall k = i, \dots, j-1) \\ &\quad + (1 - \chi_{arc}^{(M)}) \sum_{i=1}^Q \sum_{j>1} \chi_{rec}^{i,j} \chi(\sin \alpha_{jk} \geq \frac{\varepsilon^\delta}{4}, \text{ for some } k = i, \dots, j-1) \end{aligned}$$

where $\chi_{rec}^{i,j} = 1$ if and only if b_i (constructed via the sequence $t_1, \rho_1, \dots, t_i, \rho_i$) is recollided in the time interval (t_j, t_{j-1}) and α_{jk} (with $i < k < j$) is the absolute value of the sum of the angles between the outgoing velocity v_k^+ from the k -th obstacle and the recolliding velocity v_j^- , i.e.

$$\alpha_{jk} = |\varphi_k| + \sum_{r=k+1}^j |\theta_r + \varphi_r|$$

where θ_r is the deflection angle due to the r -th scatterer and φ_r is the angle covered in the time interval (t_{r+1}, t_r^+) outside the scatterers and φ_j is the angle covered between the j -th obstacle before recolliding with b_i . Here $\delta > 0$ is a suitable parameter that we will fix later. Note that, thanks to $1 - \chi_{arc}^{(M)}$, we have $|\varphi_r| \leq 2\pi/M$ for any r .

As noticed in [DP], the constraint $\sin \alpha_{jk} \leq \varepsilon^\delta/4$ implies that $|\theta_r + \varphi_r - \pi| < \varepsilon^\delta$ for some $r = i, \dots, j-1$, thus we get

$$\begin{aligned}
& \mathbb{E}_{x,v} \left[(1 - \chi_{arc}^{(M)}) \sum_{i=1}^Q \sum_{j>1} \chi_{rec}^{i,j} \chi(\sin \alpha_{jk} \leq \frac{\varepsilon^\delta}{4}, \forall k = i, \dots, j-1) \right] \\
& \leq e^{-2t\mu_\varepsilon \varepsilon^\gamma} \sum_{Q \geq 0} \mu_\varepsilon^Q \sum_{i=1}^Q \sum_{j=i+2}^Q \sum_{k=i}^{j-1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{Q-1}} dt_Q \\
& \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_1 \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_2 \cdots \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_Q \chi(|\theta_k + \varphi_k - \pi| < \varepsilon^\delta).
\end{aligned}
\tag{5.6}$$

Now we note that for M big enough

$$\begin{aligned}
\int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} d\rho_k \chi(|\theta_k + \varphi_k - \pi| < \varepsilon^\delta) &= \int_{-\pi}^{\pi} d\theta_k \check{\Gamma}_{\varepsilon,\gamma}^{(B)}(\theta_k) \chi(|\theta_k + \varphi_k - \pi| < \varepsilon^\delta) \\
&= \varepsilon \int_{-\pi}^{\pi} d\theta_k \Gamma_{\varepsilon,\gamma}^{(B)}(\theta_k) \chi(|\theta_k + \varphi_k - \pi| < \varepsilon^\delta) \\
&= \varepsilon \int_{\pi(1-\frac{2}{M})-\varepsilon^\delta}^{\pi(1-\frac{2}{M})+\varepsilon^\delta} d\theta_k \Gamma_{\varepsilon,\gamma}^{(B)}(\theta_k) \\
&\leq C\varepsilon^{1+\delta}
\end{aligned}
\tag{5.7}$$

where $\check{\Gamma}_{\varepsilon,\gamma}^{(B)}(\theta_k)$ is the differential cross section associated to the rescaled potential $\check{\psi}_\varepsilon$, while $\Gamma_{\varepsilon,\gamma}^{(B)}(\theta_k)$ is differential cross section associated to the unrescaled potential ψ_ε . In the last line of (5.7) we used that $\Gamma_{\varepsilon,\gamma}^{(B)}(\theta)$ is uniformly bounded in ε when θ is far from 0, as shown in Appendix B.

Then from (5.6) one gets

$$\begin{aligned}
& \mathbb{E}_{x,v} \left[(1 - \chi_{arc}^{(M)}) \sum_{i=1}^Q \sum_{j>1} \chi_{rec}^{i,j} \chi(\sin \alpha_{jk} \leq \frac{\varepsilon^\delta}{4}, \forall k = i, \dots, j-1) \right] \\
& \leq C e^{-2t\mu_\varepsilon \varepsilon^\gamma} \sum_{Q \geq 1} \frac{(2t\mu_\varepsilon \varepsilon^\gamma)^Q}{Q!} Q^3 \varepsilon^{1+\delta-\gamma} \leq C \varepsilon^{2\gamma-2+\delta}.
\end{aligned}
\tag{5.8}$$

For what concerns the second term in (5.5) we note that, once we fix all the variables $\{t_\ell\}_{\ell=1}^Q$ and $\{\rho_\ell\}_{\ell=1}^Q$ except t_k , by using the same geometrical argument as the one illustrated in Figure 5 of [DP], one gets that the integral over t_{k+1} is bounded by

$$\frac{2\varepsilon^\gamma}{\sin \alpha_{jk}} \leq 8\varepsilon^{\gamma-\delta}.
\tag{5.9}$$

It follows that

$$\begin{aligned}
& \mathbb{E}_{x,v} \left[(1 - \chi_{arc}) \sum_{i=1}^Q \sum_{j>1} \chi_{rec}^{i,j} \chi(\sin \alpha_{jk} \geq \frac{\varepsilon^\delta}{4}, \text{ for some } k = i, \dots, j-1) \right] \\
& \leq e^{-2t\mu_\varepsilon \varepsilon^\gamma} \sum_{Q \geq 0} \frac{(2\mu_\varepsilon \varepsilon^\gamma)^Q}{(Q-1)!} Q^3 t^{Q-1} 8\varepsilon^{\gamma-\delta} \\
& \leq CT\varepsilon^{5\gamma-\delta-4}.
\end{aligned}
\tag{5.10}$$

We can now optimize the parameter δ setting $\delta = \frac{3\gamma-2}{2}$. From equations (5.6) and (5.10) we finally end up with

$$\mathbb{E}_{x,v} [\chi_{rec}] \leq C\varepsilon^{\frac{7\gamma-6}{2}}
\tag{5.11}$$

which tends to 0 as $\varepsilon \rightarrow 0$ if $\gamma \in (6/7, 1)$. \square

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APPENDIX A. THE COLLISION TIME

We want to estimate the time spent by a the test particle in the interaction disk associated to the central potential of finite range with a uniform magnetic field perpendicular to the plane. Let be $\varepsilon^\alpha \phi(r)$ with $\alpha \in [0, 1/2)$ the central potential and εB the modulus of the magnetic field. The Lagrangian of the system is

$$\mathcal{L}(r, \dot{r}, \theta, \dot{\theta}) = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - \varepsilon^\alpha \phi(r) + \varepsilon \frac{B}{2}r^2\dot{\theta}$$

We observe that the energy of the system is conserved. Moreover the Lagrangian does not depend on the variable θ , so we obtain the conservation of the conjugate momentum

$$\frac{d}{dt}(r^2\dot{\theta} + \varepsilon \frac{B}{2}r^2) = 0.$$

Therefore we obtain the following conserved quantities

$$\begin{aligned} \dot{r}^2 + r^2\dot{\theta}^2 + 2\varepsilon^\alpha \phi &= 2E, \\ r^2\dot{\theta} + \varepsilon \frac{B}{2}r^2 &= M, \end{aligned}$$

and the equations of motion are

$$\begin{cases} \dot{r} := \frac{dr}{dt} = \sqrt{2(E - \varepsilon^\alpha \phi) - \frac{M^2}{r^2} - \frac{\varepsilon^2 B^2}{4}r^2 + \varepsilon MB} \\ \dot{\theta} := \frac{d\theta}{dt} = \frac{M}{r^2} - \varepsilon \frac{B}{2}. \end{cases}$$

This implies that

$$\begin{aligned} \frac{dt}{dr} &= \left[2(E - \varepsilon^\alpha \phi(r)) - \frac{M^2}{r^2} - \frac{\varepsilon^2 B^2}{4}r^2 + \varepsilon MB \right]^{-1/2} \\ \frac{d\theta}{dr} &= \frac{\frac{M}{r^2} - \varepsilon \frac{B}{2}}{\sqrt{2(E - \varepsilon^\alpha \phi(r)) - \frac{M^2}{r^2} - \frac{\varepsilon^2 B^2}{4}r^2 + \varepsilon MB}}. \end{aligned} \tag{A.1}$$

We now define the effective potential

$$\phi_{eff}(r) = \varepsilon^\alpha \phi(r) + \frac{M^2}{2r^2} + \frac{\varepsilon^2 B^2}{8}r^2 - \frac{\varepsilon MB}{2}.$$

and we assume that the potential has a short range, i.e. $\phi(r) : [0, 1] \rightarrow \mathbb{R}$ and ϕ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

Take the modulus of the initial velocity to be $|v| = 1$. When the particle hits the obstacle of radius $r = 1$ the conserved quantities read

$$\begin{cases} E = \frac{1}{2} \\ M = \rho + \varepsilon \frac{B}{2} \end{cases}$$

being $\rho \in [0, 1]$ is the impact parameter. The effective potential is

$$\phi_{eff}(r) = \varepsilon^\alpha \phi(r) + \left(\rho + \varepsilon \frac{B}{2} \right)^2 \frac{1}{2r^2} + \frac{\varepsilon^2 B^2}{8}r^2 - \left(\rho + \varepsilon \frac{B}{2} \right) \frac{\varepsilon B}{2} = \varepsilon^\alpha \phi(r) + \frac{1}{2} \left[\frac{\rho}{r} - \frac{\varepsilon B}{2} \left(r - \frac{1}{r} \right) \right]^2.$$

By integrating the equations of motion we obtain the collision time, namely the time spent inside the obstacle:

$$\tau = 2 \int_{r_{min}}^1 dr \left[1 - 2\varepsilon^\alpha \phi(r) - \left(\frac{\rho}{r} - \frac{\varepsilon B}{2} \left(r - \frac{1}{r} \right) \right)^2 \right]^{-1/2} \tag{A.2}$$

where r_{min} (the minimum distance from the centre) is the unique zero of the radicand, i.e.

$$1 = 2\phi_{eff}(r_{min}),$$

so we can reformulate (A.2) as

$$\tau = \sqrt{2} \int_{r_{min}}^1 \frac{dr}{\sqrt{2(\phi_{eff}(r_{min}) - \phi_{eff}(r))}}$$

where $2\phi_{eff}(r) \leq 1$. The derivative of the effective potential reads

$$\phi'_{eff}(r) = \varepsilon^\alpha \phi'(r) - \frac{\rho^2}{r^3} - \frac{\varepsilon^2 B^2}{4r^3} - \frac{\varepsilon B \rho}{r^3} + \frac{\varepsilon^2 B^2}{4} r.$$

By the mean value theorem we get

$$|\phi_{eff}(r_{min}) - \phi_{eff}(r)| = |r - r_{min}| |\phi'_{eff}(r^*)| \geq |r - r_{min}| \left(\inf_{r \in (r_{min}, 1)} |\phi'_{eff}(r)| \right), \quad r^* \in (r_{min}, r)$$

and then

$$\tau \leq \frac{\sqrt{2}}{(\inf_{r \in (r_{min}, 1)} |\phi'_{eff}(r)|)^{1/2}} \int_{r_{min}}^1 \frac{1}{\sqrt{r - r_{min}}} dr.$$

Since $\phi'_{eff}(r) < 0$ for $r \in [0, 1]$, then $\inf_{r \in (r_{min}, 1)} |\phi'_{eff}(r)| =: \kappa > 0$ and it follows easily that

$$\tau \leq 2 \left(\frac{2(1 - r_{min})}{\kappa} \right)^{1/2} \leq 2 \left(\frac{2}{\kappa} \right)^{1/2}.$$

For the corresponding rescaled problem the effective potential reads

$$(A.3) \quad \phi_{eff}^{(\varepsilon)}(r) = \varepsilon^\alpha \phi(r/\varepsilon) + \frac{1}{2} \left[\frac{\varepsilon \rho}{r} + \frac{B}{2} \left(\frac{\varepsilon^2}{r} - r \right) \right]^2$$

with $\rho \in [0, 1]$. In this way one gets $-\phi_{eff}^{(\varepsilon)'}(r) = \frac{1}{\varepsilon} F(r/\varepsilon, \varepsilon)$ where

$$(A.4) \quad F(y, \varepsilon) = -\varepsilon^\alpha \phi'(y) + (\rho + B)\rho y^{-3} + \frac{\varepsilon^2 B^2}{4y^3} (1 - y^4)$$

which is positive for $y < 1$ and uniformly in ε . The same argument as before yields the claimed estimate:

$$(A.5) \quad \tau_\varepsilon \leq \frac{(2\varepsilon)^{1/2}}{(\inf_{y \in (y_0, \varepsilon)} F(y, \varepsilon))^{1/2}} \int_{y_0}^\varepsilon \frac{dy}{\sqrt{y - y_0}} \leq C\varepsilon$$

where $y_0 = y_0(\varepsilon)$ is such that

$$1 = 2\varepsilon^\alpha \phi(y_0) + \left(\frac{\rho}{y_0} - \frac{\varepsilon^2 B y_0}{2} + \frac{B}{2y_0} \right).$$

We consider now the long range unrescaled potential defined in Assumption B1), i.e. $\psi_\varepsilon(r) = r^{-s}$ for $r < \varepsilon^{\gamma-1}$ and $\psi_\varepsilon(r) = \varepsilon^{-s(\gamma-1)}$ for $r \geq \varepsilon^{\gamma-1}$. The same argument as for the short range case leads to the following estimate for the collision time after rescaling:

$$(A.6) \quad \tau \leq \frac{(2\varepsilon)^{1/2}}{(\inf_{y \in (y_0, \varepsilon^\gamma)} \tilde{F}(y, \varepsilon))^{1/2}} \int_{y_0}^{\varepsilon^\gamma} \frac{dy}{\sqrt{y - y_0}} \leq C\varepsilon^\gamma$$

with

$$\tilde{F}(y, \varepsilon) := -\psi'_\varepsilon(y) + \varepsilon^{2(\gamma-1)} \rho^2 y^{-3} + B^2 \varepsilon^{3\gamma-2} \rho y^{-3} + B^2 y^{-3} \varepsilon^{4\gamma-2} (1 - y^4 \varepsilon^{4(1-\gamma)})/4 > 0$$

for $y \in (y_0, \varepsilon^\gamma)$ where $y_0 = y_0(\varepsilon)$ is such that $1 = 2\psi_\varepsilon(y_0) + \left[\varepsilon^{\gamma-1} \rho y_0^{-1} - \frac{B}{2} (\varepsilon^{2\gamma-1} y_0^{-1} - \varepsilon y_0) \right]^2$.

APPENDIX B. CROSS SECTION

Proposition B.1. *Consider the scattering angle $\theta(\rho, \varepsilon)$ of a particle with impact parameter ρ due to a uniform magnetic field perpendicular to the plane with modulus εB and due to a radial potential $\varepsilon^\alpha \phi$, where $\alpha > 0$ and ϕ satisfies assumptions A1, A2, A3. Consider also the scattering angle $\tilde{\theta}(\rho, \varepsilon)$ associated to the same radial potential as before, but without any magnetic field. Then, for ε small enough one gets*

$$(B.1) \quad \theta(\rho, \varepsilon) = \tilde{\theta}(\rho, \varepsilon) + O(\varepsilon).$$

Proof. Following [DR, L] we can write the exact formula for both of the scattering angles:

$$(B.2) \quad \tilde{\theta}(\rho, \varepsilon) = \pi - 2 \arcsin \rho - 2 \int_{\rho}^{\tilde{u}_{max}(\rho, \varepsilon)} \frac{du}{\sqrt{1 - u^2 - 2\varepsilon^\alpha \phi(\rho u^{-1})}}$$

where $\tilde{u}_{max}(\rho, \varepsilon)$ is the solution of the equation $\tilde{u}_{max}^2 + 2\varepsilon^\alpha \phi(\rho \tilde{u}_{max}^{-1}) = 1$, while

$$(B.3) \quad \theta(\rho, \varepsilon) = \pi - 2 \arcsin \rho - 2 \int_{\rho}^{u_{max}(\rho, \varepsilon)} du \frac{1 + \frac{\varepsilon B}{2\rho} (1 - \frac{\rho^2}{u^2})}{\sqrt{1 - 2\varepsilon^\alpha \phi(\rho u^{-1}) - u^2 [1 + \frac{\varepsilon B}{2\rho} (1 - \frac{\rho^2}{u^2})]^2}}$$

where $u_{max}(\rho, \varepsilon)$ is the solution of the equation $2\varepsilon^\alpha \phi(\rho u_{max}^{-1}) + u_{max}^2 [1 + \frac{\varepsilon B}{2\rho} (1 - \frac{\rho^2}{u_{max}^2})]^2 = 1$.

Hence, an expansion of $\theta(\rho, \varepsilon)$ for ε small enough yields the claimed asymptotic formula. \square

Proposition B.2. *Let $\tilde{\theta}$ be the scattering angle associated to the long range potential $\Psi(r) = r^{-s}$ with $s > 2$, $\theta_{\varepsilon, \gamma}$ the scattering angle due to a radial potential ψ_ε defined in Assumption B1) and $\theta_{\varepsilon, \gamma}^{(B)}$ the scattering angle due to ψ_ε and to a uniform, constant magnetic field perpendicular to the plane with modulus εB . Then one has*

- a) $\theta_{\varepsilon, \gamma}^{(B)} \rightarrow \tilde{\theta}$ as $\varepsilon \rightarrow 0$.
- b) $\Gamma_{\varepsilon, \gamma}^{(B)}(\theta) \rightarrow \Gamma(\theta)$ as $\varepsilon \rightarrow 0$, where $\Gamma_{\varepsilon, \gamma}^{(B)}(\theta)$ is the differential cross section associated to the radial potential ψ_ε and the magnetic field, while $\Gamma(\theta)$ is the one associated to the radial potential Ψ .
- c) $\Gamma_{\varepsilon, \gamma}^{(B)}(\theta) \leq C\theta^{-1-1/s}$ uniformly in ε, B .

Proof. a) Let us now consider the truncated potential $\tilde{\Psi} = r^{-s} - A^{-s}$ with $s > 2$ for $r \leq A$ and $\tilde{\Psi} = 0$ for $r > A$ with $A = \varepsilon^{\gamma-1}$ and $\gamma \in (0, 1)$. Take the modulus of the initial velocity of the light particle to be $|v| = 1$.

We denote by ρ the impact parameter (with $0 \leq \rho \leq A$) while the scattering angle (that is the angle between the ingoing and the outgoing relative velocities) is

$$(B.4) \quad \theta_{\varepsilon, \gamma}(\rho) = 2 \int_{\arcsin(\rho/A)}^{\pi/2} \left(1 - \frac{\sin \beta}{v + sv^{s-1}} \right) d\beta.$$

where $v = v(\beta)$ such that $v^2 + 2((v/\rho)^s - A^{-s}) = \sin^2 \beta$ and $v = \rho/r$ (see Appendix in [DP]). Following [DP, L], we can write the formula for the scattering angle associated to the potential $\tilde{\Psi}$ and with the uniform magnetic field. Due to its invariance under rescaling, the scattering angle associated to the equations of motion 2.4 reads

$$(B.5) \quad \begin{aligned} \theta_{\varepsilon, \gamma}^{(B)}(M) &= \pi - 2 \arcsin \left(\frac{M}{A} - \varepsilon \frac{BA^2}{2} \right) - 2 \int_{r_*}^A \frac{\frac{1}{r^2} (M - \varepsilon \frac{B}{2} r^2) dr}{\sqrt{1 - 2\tilde{\Psi}_{eff}(r)}} \\ &= 2 \int_{\arcsin(\frac{M}{A} - \frac{\varepsilon BA}{2})}^{\pi/2} d\beta \left[1 - \frac{(1 - \frac{\varepsilon BM}{2u^2}) \sin \beta}{u + su^{s-1} M^{-s} + \frac{\varepsilon^2 B^2 M^2}{4u^3}} \right] \end{aligned}$$

where $\tilde{\Psi}_{eff}(r) = \tilde{\Psi}(r) + \frac{1}{2} (\frac{M}{r} - \frac{\varepsilon}{2} Br)^2$, M is the value of the conserved momentum at the hitting time, i.e. $M = \rho + \varepsilon A^2 \frac{B}{2}$, and r_* is defined as the solution of the equation $2\tilde{\Psi}_{eff}(r_*) = 1$. In the second line we made the change $r \rightarrow u \rightarrow \beta$ where $u = u(\beta, M) = \frac{M}{r}$ and $\sin^2 \beta = 2\tilde{\Psi}(M/u)$. Note that the change of variable $u \rightarrow \beta$ is well defined because $\tilde{\Psi}(M/u)$ is non-decreasing when $u \in [M/A, M/r_*]$ for ε small enough.

From (B.4) and (B.5) it is clear that $\theta_{\varepsilon, \gamma}$ and $\theta_{\varepsilon, \gamma}^{(B)}$ have the same asymptotic behaviour as ε approaches 0. Since $\theta_{\varepsilon, \gamma} \rightarrow \tilde{\theta}$, one gets the claim.

b) The inverse of the differential cross section associated to $\tilde{\theta}$ is

$$(B.6) \quad \left| \frac{d\theta_{\varepsilon, \gamma}}{d\rho} \right| = \frac{2}{\rho^{s+1}} \int_{\arcsin(\rho/A)}^{\pi/2} \frac{d\beta \sin \beta sv^{s-1}}{(v + sv^{s-1} \rho^{-s})^2} \left[s - \frac{1 + s(s-1)v^{s-2} \rho^{-s}}{1 + sv^{s-2} \rho^{-s}} \right] + \frac{2s\rho^{-2} A^{2-s}}{1 + 2s\rho^{-2} A^{2-s}}.$$

We want to study the limit of $d\theta_{\varepsilon,\gamma}^{(B)}/d\rho$. For a mere computational convenience, we prefer to look at $d\theta_{\varepsilon,\gamma}^{(B)}/dM$ which is related to $d\theta_{\varepsilon,\gamma}^{(B)}/d\rho$ via

$$(B.7) \quad \frac{d\theta_{\varepsilon,\gamma}^{(B)}}{dM}(M) = \frac{d\theta_{\varepsilon,\gamma}^{(B)}}{d\rho}(\rho + \frac{\varepsilon B A^2}{2}).$$

From (B.5) one gets

$$(B.8) \quad \begin{aligned} \frac{d\theta_{\varepsilon,\gamma}^{(B)}}{dM} = & -\frac{2}{A\sqrt{1 - (\frac{M}{A} - \frac{\varepsilon B A}{2})^2}} \left[\frac{sM^{-2}A^{2-s} + \varepsilon B A^2 M^{-1}}{1 + sM^{-2}A^{2-s} + \frac{\varepsilon^2}{4}B^2 A^4 M^{-2}} \right] \\ & - 2 \int_{\arcsin(\frac{M}{A} - \frac{\varepsilon B A}{2})}^{\pi/2} \frac{d\beta \sin \beta}{(u + su^{s-1}M^{-s} + \frac{\varepsilon^2 B^2 M^2}{4u^3})^2} [s^2 u^{s-1} M^{-s-1} - u'(1 + s(s-1)u^{s-2}M^{-s})] \\ & - \varepsilon B M \int_{\arcsin(\frac{M}{A} - \frac{\varepsilon B A}{2})}^{\pi/2} \frac{d\beta \sin \beta}{u} \left[s(s+1)u^{s-2}M^{-s}(u'M - 1) + 3u'M \right. \\ & \left. + \frac{5\varepsilon^2 B^2 M^3 u'}{4u} + \frac{\varepsilon B M}{u^2} + \frac{3\varepsilon B M^2}{2u^3} - 1 - \frac{3\varepsilon^2 B^2 M^2}{4u^4} \right] \end{aligned}$$

where

$$(B.9) \quad u' = \frac{du}{dM} = \frac{1}{M^{s+1}} \left(\frac{su^{s-1} + (1 - \frac{\varepsilon B M}{2u^2})\frac{\varepsilon B M^{s+1}}{2u}}{1 + su^{s-2}M^{-s} - \frac{\varepsilon^2 B^2 M^2}{4u^4}} \right).$$

As for item a), one realizes that $d\theta_{\varepsilon,\gamma}^{(B)}/d\rho$ and $d\theta_{\varepsilon,\gamma}/d\rho$ are asymptotically equivalent for any ρ , thus Proposition A.1 in [DP] implies that $\Gamma_{\varepsilon,\gamma}^{(B)}(\theta) \rightarrow \Gamma(\theta)$ for $\theta \in (-\pi, \pi)$ because its inverse map converges everywhere.

c) From (B.8) for ε small enough a tedious expansion gives

$$(B.10) \quad \left| \frac{d\theta_{\varepsilon,\gamma}^{(B)}}{dM} \right| \geq \left| \frac{d\theta_{\varepsilon,\gamma}}{d\rho} \right| - \varepsilon |\mathcal{R}(B, M, \varepsilon)| \geq \frac{1}{C} \left| \frac{d\theta_{\varepsilon,\gamma}}{d\rho} \right|$$

where $C > 1$ is a constant, \mathcal{R} is bounded in ε . The claim follows thanks to Proposition A.1 in [DP]. \square

REFERENCES

- [BBS] C. Boldrighini, L. A. Bunimovich, and Y. G. Sinai. On the Boltzmann equation for the Lorentz gas. *J. Stat. Phys.* **32**, 477-501, 1983.
- [BMHH] A. V. Bobylev, F. A. Maaø, A. Hansen and E. H. Hauge. Two-Dimensional Magnetotransport according to the Classical Lorentz Model. *Phys. Rev. Lett.*, **75**, 2, 1995.
- [BMHH1] A. V. Bobylev, F. A. Maaø, A. Hansen, and E. H. Hauge. There is more to be learned from the Lorentz model. *J. Stat. Phys.* **87**, 1205-1228, 1997.
- [BHPH] A. V. Bobylev, A. Hansen, J. Piasecki and E. H. Hauge. From the Liouville equation to the generalized Boltzmann equation for magnetotransport in the 2d Lorentz model. *J. Stat. Phys.* **102**, 1133-1150, 2001.
- [BNP] G. Basile, A. Nota, and M. Pulvirenti. A diffusion limit for a test particle in a random distribution of scatterers. *J. Stat. Phys.*, **155**, Issue 6, 1087-1111, 2014.
- [BNPP] G. Basile, A. Nota, F. Pezzotti and M. Pulvirenti. Derivation of the Fick's law for the Lorentz model in a low density regime. *Commun. Math. Phys.*, **336**, Issue 3, 1607-1636, 2015.
- [DGL] D. Dürr, S. Goldstein, and J. Lebowitz. Asymptotic motion of a classical particle in a random potential in two dimensions: Landau model. *Comm. Math. Phys.* **113**, 209-230, 1987.
- [DP] L. Desvillettes and M. Pulvirenti. The linear Boltzmann equation for long-range forces: a derivation from particle systems. *Models Methods Appl. Sci.* **9**, 1123-1145, 1999.
- [DR] L. Desvillettes and V. Ricci. A rigorous derivation of a linear kinetic equation of Fokker-Planck type in the limit of grazing collisions. *J. Stat. Phys.* **104**, 1173-1189, 2001.
- [DR1] L. Desvillettes and V. Ricci. The Boltzmann-grad limit of a stochastic Lorentz Gas in a force field. *Bulletin of the Institute of Mathematics Academia Sinica (New Series)* **2**, 2, 637-648, 2007.
- [DR2] L. Desvillettes and V. Ricci. Non-Markovianity of the Boltzmann-Grad limit of a system of random obstacles in a given force field. *Bull. Sci. Math.* **128**, 1, 39-46, 2004.
- [G] G. Gallavotti. Grad-Boltzmann limit and Lorentz's Gas. *Statistical Mechanics. A short treatise*. Appendix 1.A2. Springer, Berlin, 1999.
- [H] E. H. Hauge. What can one learn from the Lorentz model? *Transport Phenomena*, G. Kircenow and J. Marro, eds., *Lecture Notes in Physics*, Vol. **12**, Springer-Verlag, Berlin, 1974.
- [K] K. Kirkpatrick. Rigorous derivation of the Landau Equation in the weak coupling limit. *Comm. Pure Appl. Math.* **8**, 1895-1916, 2009.

- [KP] H. Kesten and G. Papanicolaou. A limit theorem for stochastic acceleration. *Commun. Math. Phys.* **78**, 19-63, 1981.
- [KS] A. Kuzmany and H. Spohn. Magnetotransport in the two-dimensional Lorentz gas. *Phys. Rev. E*, **57**, 5, 1998 .
- [L] Landau, L.D., Lifshitz, E.M.: Mechanics. *Course of Theoretical Physics*. vol.1. Pergamon press, Oxford, 1960.
- [S] H. Spohn. The Lorentz flight process converges to a random flight process. *Comm. Math. Phys.* **60**, 277-290, 1978.

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